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Fachbereich Mathematik

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Abstract

A logarithmic type Lieb-Thirring inequality for two-dimensional Schrödinger operators is established. The result is applied to prove spectral estimates on trapped modes in quantum layers.

1   Introduction

It is well known that the sum of the moments of negative eigenvalues $-\lambda_j$ of a one-dimensional Schrödinger operator $-\frac{d^2}{dx^2} - V$ can be estimated by

$$\sum_j \lambda_j^\gamma \ \leq \ L_{\gamma,1} \int_{\mathbb{R}} V_+(x)^{\gamma + \frac{1}{2}} \, dx, \quad \gamma \geq \frac{1}{2}, \tag{1}$$

where $L_{\gamma,1}$ is a constant independent of V, see [8], [12]. For $\gamma = \frac{1}{2}$ this bound has the correct weak coupling behavior, see [10], and it also shows the correct Weyl-type asymptotics in the semi-classical limit. Moreover, (1) fails to hold whenever $\gamma < \frac{1}{2}$. The case $\gamma = \frac{1}{2}$ therefore represents certain borderline inequality in dimension one.

The situation is much less satisfactory in dimension two. The corresponding two-dimensional Lieb-Thirring bound

$$\sum_j \lambda_j^\gamma = \text{tr}\,(-\Delta - V)_-^\gamma \ \leq \ L_{\gamma,2} \int_{\mathbb{R}^2} V_+(x)^{\gamma+1} \, dx \tag{2}$$

holds for all $\gamma > 0$, [8]. Dimensional analysis shows that here the borderline should be $\gamma = 0$. However, (2) fails for $\gamma = 0$, because $-\Delta - V$ has at least one negative eigenvalue whenever $\int V \geq 0$, see [10]. In addition, it was shown in [10] that if V decays fast enough, the operator $-\Delta - \alpha V$ has for small α only one eigenvalue which goes to zero exponentially fast:

$$\lambda_1 \sim e^{-4\pi(\alpha \int V)^{-1}}, \quad \alpha \to 0. \tag{3}$$

It follows from (3) that the optimal behavior for $\alpha \to 0$ cannot be reached in the power-like scale (2), no matter how small γ is, since the l.h.s. decays faster than any power of α. This means that in order to obtain a Lieb-Thirring type inequality with the optimal behavior in the weak coupling limit, one should introduce a different scale on the l.h.s. of (2).

In the present paper we want to find a two-dimensional analog of the one-dimensional borderline inequality, which corresponds to $\gamma = \frac{1}{2}$ in (1). In other words, we want to establish an inequality with the r.h.s. proportional to V and with the correct order of asymptotics in weak and strong coupling regime. Obviously, we have to replace the power function on the l.h.s. of (2) by a new function $F(\lambda)$, which will approximate identity as close as possible. On the other hand, since $-\Delta - V$ has always at least one eigenvalue, it is necessary that $F(0) = 0$. Moreover, equation (3) shows that F should grow from zero faster than any power of λ, namely as $|\ln \lambda|^{-1}$. This leads us to define the family of functions $F_s : (0, \infty) \to (0, 1]$ by

$$\forall\, s > 0 \qquad F_s(t) := \begin{cases} |\ln ts^2|^{-1} & 0 < t \leq e^{-1}s^{-2}, \\[2mm] 1 & t > e^{-1}s^{-2}. \end{cases} \tag{4}$$

Notice that each F_s is non decreasing and continuous and that $F_s(t) \to 1$ point-wise as $s \to \infty$. Hence our goal is to establish an appropriate estimate on the regularized counting function $\sum_j F_s(\lambda_j)$ for large values of the parameter s.

Our main results is formulated in the next section. It turns out, that $\sum_j F_s(\lambda_j)$ can be estimated by a sum of two integrals, one of which includes a local logarithmic weight, see Theorem 1. The inequality (8) established in Theorem 1 has the correct behavior for weak as well as for

5

strong potentials, see Remark 1. We also show that the logarithmic weight in (8) cannot be removed, see Remark 2. Moreover, in Corollary 1 we obtain individual estimates on eigenvalues of Schrödinger operators with slowly decaying potentials. The proof of the main result, including two auxiliary Lemmata, is then given in section 3. In the closing section 4 we apply Theorem 1 to analyze discrete spectrum of a Schrödinger operators corresponding to quantum layers. The result established in section 4 may be regarded as two-dimensional analog of Lieb-Thirring inequalities on trapped modes in quantum waveguides obtained in [5].

2   Main results

For a given V we define the Schrödinger operator

$$-\Delta - V \quad \text{in} \quad L^2(\mathbb{R}^2) \tag{5}$$

as the Friedrich extension of the operator associated with the quadratic form

$$Q_V[u] = \int_{\mathbb{R}^2} \left(|\nabla u|^2 - V|u|^2 \right) dx \quad \text{on} \quad C_0^\infty(\mathbb{R}^2), \tag{6}$$

provided Q_V is bounded from below. Throughout the paper we will suppose that V satisfies

Assumption A. The function $V(x)$ is such that $\sigma_{ess}(-\Delta - V) = [0, \infty)$.

Following notation will be used in the text. Given a self-adjoint operator T, the number of negative eigenvalues, counting their multiplicity, of T to the left of a point $-\nu$ is denoted by $N(\nu, T)$. The symbol $\mathbb{R}_+$ stands for the set $(0, \infty)$. Moreover, as in [6] we define the space $L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))$ in polar coordinates (r, θ) in $\mathbb{R}^2$, as the space of functions f such that

$$\|f\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))} := \int_0^\infty \left(\int_0^{2\pi} |f(r, \theta)|^p \, d\theta \right)^{1/p} r \, dr < \infty. \tag{7}$$

Finally, given $s > 0$ we denote $B(s) := \{ x \in \mathbb{R}^2 : |x| < s \}$. We then have

Theorem 1. *Let* $V \geq 0$ *and* $V \in L^1_{loc}(\mathbb{R}^2, |\ln |x|| \, dx)$. *Assume that* $V \in L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))$ *for some* $p > 1$. *Then the quadratic form* (6) *is bounded from below and closable. The negative eigenvalues* $-\lambda_j$ *of the operator associated with its closure satisfy the inequality*

$$\sum_j F_s(\lambda_j) \leq c_1 \|V \ln(|x|/s)\|_{L^1(B(s))} + c_p \|V\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))} \tag{8}$$

for all $s \in \mathbb{R}_+$. *The constants* c_1 *and* c_p *are independent of* s *and* V.
In particular, if $V(x) = V(|x|)$, *then there exists a constant* C, *such that*

$$\sum_j F_s(\lambda_j) \leq C \left(\|V \ln(|x|/s)\|_{L^1(B(s))} + \|V\|_{L^1(\mathbb{R}^2)} \right) \tag{9}$$

holds true for all $s \in \mathbb{R}_+$.

Remark 1. Notice that the r.h.s. of (8) has the right order of asymptotics in both weak and strong coupling limits. Indeed, replacing V by αV and assuming that $V \in L^1(\mathbb{R}^2, (|\ln |x|| + 1) \, dx)$ it can be seen from the definition of F_s that

$$\sum_j F_s(\lambda_j) \sim \alpha, \quad \alpha \to 0 \ \vee \ \alpha \to \infty.$$

6

For $\alpha \to 0$ this follows from (3). For $\alpha \to \infty$ is the behavior of $\sum_j F_s$ governed by the Weyl asymptotics for the counting function:

$$N(e^{-1}s^{-2}, -\Delta - \alpha V) \leq \sum_j F_s(\lambda_j) \leq N(0, -\Delta - \alpha V). \tag{10}$$

The latter is linear in α when $\alpha \to \infty$ provided $V \in L^1(\mathbb{R}^2, (|\ln|x|| + 1)\,dx)$, see also Remark 4.

Remark 2. We would like to emphasize that $\sum_j F_s(\lambda_j)$ cannot be estimated only in terms of $\|V\|_{L^1(\mathbb{R}^2)}$. In particular, the logarithmic term in (8) and (9) cannot be removed. This is due to the fact that there exist potentials $V \in L^1(\mathbb{R}^2)$ with a strong local singularity, such that the semi-classical asymptotics of $N(\nu, -\Delta - V)$ is non-Weyl for any $\nu > 0$, [2]. Namely if we define

$$
\begin{aligned}
V_\sigma(x) &= r^{-2} |\ln r|^{-2} |\ln|\ln r||^{-1/\sigma}, \quad r < e^{-2}, \quad \sigma > 1 \\
V_\sigma(x) &= 0, \quad r \geq e^{-2},
\end{aligned} \tag{11}
$$

where $r = |x|$, then $V_\sigma \in L^1(\mathbb{R}^2)$ for all $\sigma > 1$, but

$$N(\nu, -\Delta - \alpha V_\sigma) \sim \alpha^\sigma \qquad \alpha \to \infty, \quad \forall\, \nu > 0, \tag{12}$$

see [2, Sec. 6.5]. If (9) were true with the logarithmic factor removed, it would be in obvious contradiction with (10) and (12). Moreover, the asymptotics (12) remains valid also if the singularity of V is not placed at zero, but some other point. This shows that the condition $p > 1$ in Theorem 1 is necessary.

Remark 3. The non-Weyl asymptotics of $N(0, -\Delta - \alpha V)$ can also occur for potentials which have no singularities, but which decay at infinity too slowly, so that the associated eigenvalues accumulate at zero. For example, if

$$
\begin{aligned}
V_\sigma^\Phi(x) &= \Phi(\theta)\, r^{-2} (\ln r)^{-2} (\ln\ln r)^{-1/\sigma}, \quad r > e^2, \quad \sigma > 1 \\
V_\sigma^\Phi(x) &= 0, \quad r \leq e^2,
\end{aligned} \tag{13}
$$

then

$$N(0, -\Delta - \alpha V_\sigma^\Phi) \sim \alpha^\sigma,$$

see [2]. In this case, however, Theorem 1 says that the eigenvalues accumulating at zero are small enough so that their total contribution to $\sum_j F_s(\lambda_j)$ grows at most linearly in α. More exactly, inequality (8) gives the following estimate:

Corollary 1. *Let* $\Phi \in L^p(0, 2\pi)$ *for some* $p > 1$. *Let* V *satisfy the assumptions of Theorem 1 and suppose that*

$$V(x) - V_\sigma^\Phi(x) = o\left(V_\sigma^{|\Phi|}(x)\right), \quad |x| \to \infty,$$

where $V_\sigma^\Phi(x)$ *is defined by* (13). *Denote* $n(\alpha) = N(0, -\Delta - \alpha V)$ *and let* $-\lambda_{n(\alpha)}$ *be the largest eigenvalue of* $-\Delta - \alpha V$. *Then, for any fixed* $s > 0$ *there exists a constant* $c_s > 0$ *such that for* α *large enough*

$$\lambda_{n(\alpha)} \leq s^{-2} \exp(-c_s\, \alpha^{\sigma-1}). \tag{14}$$

Proof. Inequality (8) shows that $\sum_j F_s(\lambda_j) \leq c_s'\alpha$ for some c_s'. In particular, this implies

$$j\, F_s(\lambda_j) \leq c_s'\, \alpha, \quad \forall j. \tag{15}$$

On the other hand, from [2, Prop. 6.1] follows that $n(\alpha) \geq \tilde{c}\alpha^\sigma$ for some $\tilde{c}$ and α large enough. An application of the inequality (15) with $j = n(\alpha)$ then yields (14). Analogous estimates for $\lambda_{n(\alpha)-k}$, $k \in \mathbb{N}$ can be obtained by an obvious modification. $\square$

7

3   Proof of Theorem 1

We prove the inequality (8) for continuous potentials with compact support. The general case then follows by approximating V by a sequence of continuous compactly supported functions and using a standard limiting argument in (8).

As usual in the borderline situations, the method of [8] cannot be directly applied and a different strategy is needed. We shall treat the operator $-\Delta - V$ separately on the space of spherically symmetric functions in $L^2(\mathbb{R}^2)$ and on its orthogonal complement. To this end we define the corresponding projection operators:

$$(Pu)(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta)\, d\theta\,, \quad Qu = u - Pu\,, \quad u \in L^2(\mathbb{R}^2)\,.$$

Since P and Q commute with $-\Delta$, the variational principle says that for each $a > 1$ the operator inequality

$$-\Delta - V \geq P\left(-\Delta - (1 + a^{-1})\, V\right) P + Q\left(-\Delta - (1 + a)\, V\right) Q \tag{16}$$

holds. Let us denote by $-\lambda_j^P$ and $-\lambda_j^Q$ the non decreasing sequences of negative eigenvalues of the operators $P\left(-\Delta - (1 + a^{-1}) V\right) P$ and $Q\left(-\Delta - (1 + a V) Q\right.$ respectively. Clearly we have

$$\sum_j F_s(\lambda_j) \leq \sum_j F_s(\lambda_j^P) + \sum_j F_s(\lambda_j^Q)\,. \tag{17}$$

We are going to find appropriate bounds on the two terms on the r.h.s. of (17) separately. First we note that $P\left(-\Delta - (1 + a^{-1}) V\right) P$ is unitarily equivalent to the operator

$$h = -\frac{d^2}{dr^2} - \frac{1}{4r^2} - W(r) = h_0 - W(r) \quad \text{in} \quad L^2(\mathbb{R}_+) \tag{18}$$

with the Dirichlet boundary condition at zero and with the potential

$$W(r) = \frac{1 + a}{2\pi a} \int_0^{2\pi} V(r,\theta)\, d\theta\,. \tag{19}$$

More precisely, h is associated with the closure of the quadratic form

$$q[\varphi] = \int_{\mathbb{R}_+} \left(|\varphi'|^2 - W|\varphi|^2\right) r\, dr \quad \text{on} \quad C_0^\infty(\mathbb{R}_+)\,. \tag{20}$$

We start with the estimate on the lowest eigenvalue of h.

Lemma 1. *Let V be continuous and compactly supported and let W be given by (19). Denote by $-\lambda_1^P$ the lowest eigenvalue of the operator h. Then there exists a constant c_2, independent of s, such that*

$$F_s(\lambda_1^P) \leq c_2 \int_0^\infty W(r)\, r \left(1 + \chi_{(0,s)}(r)\, |\ln r/s|\right)\, dr\,. \tag{21}$$

holds true for all $s \in \mathbb{R}_+$.

Proof. From the Sturm-Liouville theory we find the Green function of the operator h_0 at the point $-\kappa^2$:

$$G_0(r,r',\kappa) := \begin{cases} \sqrt{rr'}\, I_0(\kappa r)\, K_0(\kappa r') & 0 \leq r \leq r' < \infty, \\[2mm] \sqrt{rr'}\, I_0(\kappa r')\, K_0(\kappa r) & 0 \leq r' < r < \infty, \end{cases}$$

where I_0, K_0 are the modified Bessel functions, see [1]. The Birman-Schwinger principle tells us that if for certain value of κ the trace of the operator

$$K(\kappa) := \sqrt{W}\, (h_0 + \kappa^2)^{-1}\, \sqrt{W}$$

8

is less than or equal to 1, then the inequality $\lambda_1^P \leq \kappa^2$ holds. Taking into account the continuity of W, this implies

$$\int_0^\infty r \, I_0\left(\sqrt{\lambda_1^P}\, r\right) K_0\left(\sqrt{\lambda_1^P}\, r\right) W(r) \, dr \geq 1. \tag{22}$$

Now we introduce the substitutions $\tau = s\sqrt{\lambda_1^P}$, $t = s^{-1}r$ and recall that $I_0(0) = 1$ while K_0 has a logarithmic singularity at zero, see [1, Chap.9]. We thus find out that

$$F_1\left(\tau^2\right) I_0(\tau t) K_0(\tau t) \leq c_2 \left(1 + \chi_{(0,1)}(t)\,|\ln t|\right), \quad \forall \tau \geq 0,$$

where c_2 is a suitable constant independent of τ. Here we have used the fact that

$$|I_0(z)\, K_0(z)| \leq \text{const} \quad \forall z \geq 1, \tag{23}$$

see [1]. Finally, we multiply both sides of inequality (22) by $F_s(\lambda_1^P)$ and note that

$$F_s(\lambda_1^P) = F_s\left(\tau^2/s^2\right) = F_1\left(\tau^2\right).$$

The proof is complete. $\qquad\square$

Next we estimate the higher eigenvalues of h.

Lemma 2. *Under the assumptions of Lemma 1 there exists a constant c_3 such that*

$$\sum_{j\geq 2} F_s(\lambda_j^P) \leq \int_0^s W(r)\, r \, |\ln r/s| \, dr + c_3 \int_s^\infty W(r)\, r \, dr, \quad \forall s \in \mathbb{R}_+.$$

Proof. Let us introduce the auxiliary operator

$$h_d = -\frac{d^2}{dr^2} - \frac{1}{4r^2} - W(r) \quad \text{in} \quad L^2(\mathbb{R}_+) \tag{24}$$

subject to the Dirichlet boundary conditions at zero and at the point s. Let $-\mu_j$ be the non decreasing sequence of negative eigenvalues of h_d. Since imposing the Dirichlet boundary condition at s is a rank one perturbation, it follows from the variational principle that

$$\sum_{j\geq 2} F_s(\lambda_j^P) \leq \sum_{j\geq 1} F_s(\mu_j). \tag{25}$$

Moreover, h_d is unitarily equivalent to the orthogonal sum $h_1 \oplus h_2$, where

$$h_1 = h_{1,0} - W(r) = -\frac{d^2}{dr^2} - \frac{1}{4r^2} - W(r) \quad \text{in} \quad L^2(0,s)$$

$$h_2 = h_{2,0} - W(r) = -\frac{d^2}{dr^2} - \frac{1}{4r^2} - W(r) \quad \text{in} \quad L^2(s,\infty)$$

with Dirichlet boundary conditions at 0 and s. Keeping in mind that $F_s \leq 1$ we will estimate (25) as follows:

$$\sum_j F_s(\mu_j) \leq N(0, h_1) + \sum_j F_s(\mu_j'), \tag{26}$$

where $-\mu_j'$ are the negative eigenvalues of h_2. To continue we calculate the diagonal elements of the Green functions of the free operators $h_{1,0}$ and $h_{2,0}$. Similarly as in the proof of Lemma 1 we get

$$
\begin{aligned}
G_1(r,r,\kappa) &= r\, I_0(\kappa r)\left(K_0(\kappa r) + \beta_s^{-1}(\kappa)I_0(\kappa r)\right) & 0 \leq r \leq s \\
G_2(r,r,\kappa) &= r\, K_0(\kappa r)\left(I_0(\kappa r) + \beta_s(\kappa)K_0(\kappa r)\right) & s \leq r < \infty,
\end{aligned}
\tag{27}
$$

9

where
$$\beta_s(\kappa) = -\frac{I_0(\kappa s)}{K_0(\kappa s)}.$$

The Birman-Schwinger principle thus gives us the following estimates on the number of eigenvalues of h_1 and h_2 to the left of the point $-\kappa^2$:

$$N(\kappa^2, h_1) \leq \int_0^s G_1(r, r, \kappa) W(r) \, dr, \quad N(\kappa^2, h_2) \leq \int_s^\infty G_2(r, r, \kappa) W(r) \, dr. \tag{28}$$

Passing to the limit $\kappa \to 0$ and using the asymptotic behavior of the Bessel functions I_0 and K_0, [1], we find out that for any fixed r holds the identity

$$\lim_{\kappa \to 0} G_1(r, r, \kappa) = \lim_{\kappa \to 0} G_2(r, r, \kappa) = r \, |\ln r/s| \tag{29}$$

The assumption on W and the dominated convergence theorem then allow us to interchange the limit $\kappa \to 0$ with the integration in (28) to obtain

$$N(0, h_1) \leq \int_0^s r \, |\ln r/s| \, W(r) \, dr. \tag{30}$$

This estimates the first term in (26). In order to find an upper bound on the second term in (26), we employ the formula

$$\sum_j F_s(\mu_j') = \int_0^\infty F_s'(t) \, N(t, h_2) \, dt, \tag{31}$$

see [8]. Using (28), the substitution $t \to t^2$ and the Fubini theorem we get

$$\sum_j F_s(\mu_j') \leq \frac{1}{2} \int_s^\infty W(r) \int_0^{e^{-1/2}s^{-1}} \frac{G_2(r, r, t)}{t \, (\ln ts)^2} \, dt \, dr.$$

In view of (27) it suffices to show that the integral

$$\int_0^{e^{-1/2}s^{-1}} \frac{K_0(tr) \, (I_0(tr) + \beta_s(t)K_0(tr))}{t \, (\ln ts)^2} \, dt \tag{32}$$

is uniformly bounded for all $s > 0$ and $r \geq s$. The substitutions $r = sy$, $t = \tau/s$ transform (32) into

$$g(y) := \int_0^{e^{-1/2}} \frac{K_0(\tau y) \, (I_0(\tau y) + \beta_1(\tau)K_0(\tau y))}{\tau \, (\ln \tau)^2} \, d\tau, \quad y \in [1, \infty). \tag{33}$$

Since g is continuous, due to the continuity of Bessel functions, and $g(1) = 0$, it is enough to check that $g(y)$ remains bounded as $y \to \infty$. Moreover, the inequality

$$(u, (h_{2,0} + t_1)^{-1} u) \leq (u, (h_{2,0} + t_2)^{-1} u) \quad \forall \, 0 \leq t_2 \leq t_1, \forall u \in L^2(s, \infty)$$

shows that $G_2(r, r, t)$, the diagonal element of the integral kernel of $(h_{2,0} + t^2)^{-1}$, is non increasing in t for each $r \geq s$. Equations (27) and (29) then imply

$$\int_0^{y^{-1}} \frac{K_0(\tau y) \, (I_0(\tau y) + \beta_1(\tau)K_0(\tau y))}{\tau \, (\ln \tau)^2} \, d\tau \leq \ln y \int_0^{y^{-1}} \frac{d\tau}{\tau \, (\ln \tau)^2} = 1.$$

On the other hand, when $\tau \in [y^{-1}, e^{-1/2}]$, it can be seen from (23) and from the behavior of I_0, K_0 in the vicinity of zero, see [1], that

$$|K_0(\tau y) \, (I_0(\tau y) + \beta_1(\tau)K_0(\tau y))| \leq \text{const}$$

uniformly in y. Equation (31) thus yields

$$\sum_j F_s(\mu_j') \leq c_3 \int_s^\infty W(r) \, r \, dr \quad \forall \, s \in \mathbb{R}_+,$$

where c_3 is independent of s. Together with (25), (26) and (30) this completes the proof. $\square$

From equation (19), Lemma 1 and Lemma 2 we conclude that

$$\sum_j F_s(\lambda_j^P) \leq (c_2 + 1)\,\|V \ln(|x|/s)\|_{L^1(B(s))} + c_3\,\|V\|_{L^1(\mathbb{R}^2)}\,.$$

Let us now turn to the second term on the r.h.s. of (17). The key ingredient in estimating this contribution will be the result of [6]. We make use of the estimate

$$\sum_j F_s(\lambda_j^Q) \leq N\left(0, Q(-\Delta - (1+a)\,V)Q\right)$$

and of the Hardy-type inequality

$$Q\,(-\Delta)\,Q \geq Q\,\frac{1}{|x|^2}\,Q\,, \tag{34}$$

which holds in the sense of quadratic forms on $C_0^\infty(\mathbb{R}^2)$, see [2]. For any $\varepsilon \in (0,1)$ we thus get the lower bound

$$Q\,(-\Delta - (1+a)\,V)\,Q \geq (1-\varepsilon)\,Q\left(-\Delta + \frac{\varepsilon}{1-\varepsilon}\,\frac{1}{|x|^2} - \frac{1+a}{1-\varepsilon}\,V\right)Q\,, \tag{35}$$

which implies

$$N\left(0, Q\,(-\Delta - (1+a)\,V)\,Q\right) \leq N\left(0, -\Delta + \frac{\varepsilon}{1-\varepsilon}\,\frac{1}{|x|^2} - \frac{1+a}{1-\varepsilon}\,V\right)\,. \tag{36}$$

The last quantity can be estimated using [6, Thm.1.2], which says that

$$N\left(0, -\Delta + \frac{\varepsilon}{1-\varepsilon}\,\frac{1}{|x|^2} - \frac{1+a}{1-\varepsilon}\,V\right) \leq \tilde{c}_p\,\|V\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))}\,. \tag{37}$$

for some constant $\tilde{c}_p$ that also depends on ε and a. In order to conclude the proof of (8) we note that by the Hölder inequality

$$\|V\|_{L^1(\mathbb{R}^2)} \leq \text{const}\,\|V\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))}\,.$$

To show that the quadratic form (6) is semi-bounded from below we note that inequality (8) says that there are only finitely many eigenvalues of $-\Delta - V$ below $-e^{-1}\,s^{-2}$. Let $-\Lambda_V$ be the minimum of those. Then

$$Q_V[u] \geq -\Lambda_V\,\|u\|_{L^2(\mathbb{R}^2)} \quad \forall\,u \in C_0^\infty(\mathbb{R}^2)\,.$$

The proof of Theorem 1 is now complete.

Remark 4. As a corollary of the proof of Theorem 1 we immediately obtain

$$N(0, -\Delta - V) \leq 1 + \text{const}\left(\|V \ln|x|\|_{L^1(\mathbb{R}^2)} + \|V\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))}\right)\,, \tag{38}$$

which agrees with [11, Thm.3].

Remark 5. Lieb-Thirring inequalities for the operator $h = h_0 - W$ in the form

$$\text{tr}\,(h_0 - W)_-^\gamma \leq C_{\gamma,a}\int_{\mathbb{R}_+} W(r)_+^{\gamma + \frac{1+a}{2}}\,r^a\,dr\,, \quad \gamma > 0\,, a \geq 1$$

have been recently established in [4].

4 Applications

In this section we consider a model of quantum layers. It concerns a conducting plate $\Omega = \mathbb{R}^2 \times (0, d)$ with an electric potential V. We will consider the shifted Hamiltonian

$$H_V = -\Delta_\Omega - V - \frac{\pi^2}{d^2} \quad \text{in} \quad L^2(\Omega), \tag{39}$$

with Dirichlet boundary conditions at $\partial\Omega$, which is associated with the closed quadratic form

$$\int_\Omega \left(|\nabla u|^2 - V|u|^2 - \frac{\pi^2}{d^2} |u|^2 \right) dx \quad \text{on} \quad H_0^1(\Omega). \tag{40}$$

We assume that for each $x_3 \in (0, d)$ the function $V(\cdot, \cdot, x_3)$ satisfies Assumption A. Without loss of generality we assume that $V \geq 0$, otherwise we replace V by its positive part.

The essential spectrum of the Operator H_V covers the half line $[0, \infty)$. Let us denote by $-\tilde{\lambda}_j$ the non decreasing sequences of negative eigenvalues of H_V. For the sake of brevity we choose $s = 1$ and prove

Theorem 2. *Assume that $V \in L^{3/2}(\Omega)$ and that*

$$\tilde{V}(x_1, x_2) = \frac{2}{d} \int_0^d V(x_1, x_2, x_3) \sin^2 \left(\frac{\pi x_3}{d} \right) dx_3$$

satisfies the assumptions of Theorem 1 for some $p > 1$. Then there exist positive constants $C_1, C_2, C_3(p)$ such that

$$\sum_j F_1(\tilde{\lambda}_j) \leq C_1 \|\tilde{V} \ln(x_1^2 + x_2^2)\|_{L^1(B(1))} + C_3(p) \|\tilde{V}\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))}$$

$$+ C_2 \|V^{3/2}\|_{L^1(\Omega)}. \tag{41}$$

Remark 6. Notice that (41) has the right asymptotic behavior in both weak and strong coupling limits. Namely, in the weak coupling limit the r.h.s. is dominated by the term linear in V, while in the strong coupling limit prevails the term proportional to $V^{3/2}$. In this sense our result is similar to the Lieb-Thirring inequalities on trapped modes in quantum wires obtained in [5].

Proof of Theorem 2. Let $\nu_k = k^2 \pi^2 / d^2$, $k \in \mathbb{N}$ be the eigenvalues of the Dirichlet Laplacian on $(0, d)$ associated with the normalized eigenfunctions

$$\phi_k(x_3) = \sqrt{\frac{2}{d}} \sin \left(\frac{k \pi x_3}{d} \right).$$

Moreover, define

$$R = (\phi_1, \cdot) \phi_1, \quad S = \mathbb{I} - R.$$

By the same variational argument used in the previous section we obtain the inequality

$$H_V \geq R \left(-\Delta_\Omega - \nu_1 - 2V \right) R + S \left(-\Delta_\Omega - \nu_1 - 2V \right) S. \tag{42}$$

The latter implies

$$\sum_j F_1(\tilde{\lambda}_j) \leq \sum_j F_1(\tilde{\mu}_j) + N(0, S \left(-\Delta_\Omega - \nu_1 - 2V \right) S), \tag{43}$$

where $-\tilde{\mu}_j$ are the negative eigenvalues of $R \left(-\Delta_\Omega - \nu_1 - 2V \right) R$. Since

$$R \left(-\Delta_\Omega - \nu_1 - 2V \right) R = (-\partial_{x_1}^2 - \partial_{x_2}^2 - 2\tilde{V}) \otimes R,$$

12

the first term on the r.h.s. of (43) can be estimated using (8) as follows:

$$\sum_j F_1(\tilde{\mu}_j) \leq C_1 \|\tilde{V}_1 \ln(x_1^2 + x_2^2)\|_{L^1(\mathbb{R}^2)} + C_3(p) \|\tilde{V}\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))}. \tag{44}$$

As for the second term, we note that

$$S\left(-\partial_{x_3}^2 - \nu_1\right)S = \sum_{k=2}^{\infty}(\nu_k - \nu_1)\left(\phi_k, \cdot\right)\phi_k \geq \sum_{k=2}^{\infty}\frac{\nu_2 - \nu_1}{\nu_2}\,\nu_k\left(\phi_k, \cdot\right)\phi_k$$
$$= \frac{3}{4}\,S\left(-\partial_{x_3}^2\right)S$$

holds true in the sense of quadratic forms on $C_0^\infty(0, d)$, which implies the estimate

$$S\left(-\Delta_\Omega - \nu_1 - 2V\right)S \geq \frac{3}{4}\,S\left(-\Delta_\Omega - \frac{8}{3}\,V\right)S.$$

Using the variational principle and the Cwickel-Lieb-Rosenblum inequality, [3, 7, 9], we thus arrive at

$$N(0, S\left(-\Delta_\Omega - \nu_1 - 2V\right)S) \leq N\left(0, -\Delta_\Omega - \frac{8}{3}\,V\right) \leq C_2 \int_\Omega V^{3/2}.$$

In view of (43) this concludes the proof.

$\square$

Acknowledegement

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Hynek Kovař́ı k
Faculty of Mathematics and Physics
Stuttgart University
D-705 69 Stuttgart, Germany

Semjon Vugalter
Faculty of Mathematics and Physics
Stuttgart University
D-705 69 Stuttgart, Germany

Timo Weidl
Faculty of Mathematics and Physics
Stuttgart University
D-705 69 Stuttgart, Germany

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