

**Universität  
Stuttgart**

**Fachbereich  
Mathematik**

---

**Stacked polytopes and tight triangulations of manifolds**

Felix Effenberger

---

**Preprint 2010/012**



**Universität  
Stuttgart**

**Fachbereich  
Mathematik**

---

Stacked polytopes and tight triangulations of manifolds

Felix Effenberger

---

**Preprint 2010/012**

Fachbereich Mathematik  
Fakultät Mathematik und Physik  
Universität Stuttgart  
Pfaffenwaldring 57  
D-70 569 Stuttgart

**E-Mail:** [preprints@mathematik.uni-stuttgart.de](mailto:preprints@mathematik.uni-stuttgart.de)  
**WWW:** <http://www.mathematik.uni-stuttgart.de/preprints>

ISSN **1613-8309**

© Alle Rechte vorbehalten. Nachdruck nur mit Genehmigung des Autors.  
L<sup>A</sup>T<sub>E</sub>X-Style: Winfried Geis, Thomas Merkle

## Abstract

Tightness of a triangulated manifold is a topological condition, roughly meaning that any simplexwise linear embedding of the triangulation into Euclidean space is “as convex as possible”. It can thus be understood as a generalization of the concept of convexity. In even dimensions, super-neighborliness is known to be a purely combinatorial condition which implies the tightness of a triangulation. Here we present other sufficient and purely combinatorial conditions which can be applied to the odd-dimensional case as well. One of the conditions is that all vertex links are stacked spheres, which implies that the triangulation is in Walkup’s class  $\mathcal{K}(d)$ . We show that in any dimension  $d \geq 4$  *tight-neighborly* triangulations as defined by Lutz, Sulanke and Swartz are tight. Furthermore, triangulations with  $k$ -stacked vertex links and the centrally symmetric case are discussed.

**Keywords:** triangulated manifold, stacked polytope, tight, perfect Morse function

**MSC:** primary 52B05, secondary 52B70, 53C42, 52B70, 57Q35

## 1 Introduction and results

Tightness is a notion developed in the field of differential geometry as the equality of the (normalized) *total absolute curvature* of a submanifold with the lower bound *sum of the Betti numbers* [42, 10]. It was first studied by Alexandrov [1], Milnor [49], Chern and Lashof [17] and Kuiper [41] and later extended to the polyhedral case by Banchoff [8], Kuiper [42] and Khnel [36].

From a geometrical point of view, tightness can be understood as a generalization of the concept of convexity that applies to objects other than topological balls and their boundary manifolds since it roughly means that an embedding of a submanifold is “as convex as possible” according to its topology. The usual definition is the following:

**Definition 1.1 (tightness [42, 36])** *Let  $\mathbb{F}$  be a field. An embedding  $M \rightarrow \mathbb{E}^N$  of a compact manifold is called  $k$ -tight with respect to  $\mathbb{F}$  if for any open or closed halfspace  $h \subset \mathbb{E}^N$  the induced homomorphism*

$$H_i(M \cap h; \mathbb{F}) \longrightarrow H_i(M; \mathbb{F})$$

*is injective for all  $i \leq k$ .  $M$  is called  $\mathbb{F}$ -tight if it is  $k$ -tight for all  $k$ . The standard choice for the field of coefficients is  $\mathbb{F}_2$  and an  $\mathbb{F}_2$ -tight embedding is called tight.*

With regard to PL embeddings of PL manifolds tightness of *combinatorial manifolds* can also be defined via a purely combinatorial condition as follows. For an introduction to PL topology see [58], for more recent developments in the field see [44, 20].

**Definition 1.2 (combinatorial manifold, combinatorial tightness [36])**

- (i) *A simplicial complex  $K$  that has a topological manifold as its underlying set  $|K|$  is called triangulated manifold.  $K$  is called combinatorial manifold of dimension  $d$  if all vertex links of  $K$  are PL  $(d-1)$ -spheres, where a PL  $(d-1)$ -sphere is a triangulation of the  $(d-1)$ -sphere that carries a standard PL structure.*
- (ii) *Let  $\mathbb{F}$  be a field. A combinatorial manifold  $K$  on  $n$  vertices is called  $(k-)$ tight w.r.t.  $\mathbb{F}$  if its canonical embedding  $K \subset \Delta^{n-1} \subset \mathbb{E}^{n-1}$  is  $(k-)$ tight w.r.t.  $\mathbb{F}$ , where  $\Delta^{n-1}$  denotes the  $(n-1)$ -dimensional simplex.*

In dimension  $d = 2$  the following are equivalent for a triangulated surface  $S$  on  $n$  vertices: (i)  $S$  has a complete edge graph  $K_n$ , (ii)  $S$  appears as a so called *regular case* in Heawood’s Map Color Theorem [27, 57], compare [36, Chap. 2C] and (iii) the induced piecewise linear embedding of  $S$  into Euclidean  $(n-1)$ -space has the two-piece property [9], and it is tight [32], [36, Chap. 2D].

Kühnel investigated the tightness of combinatorial triangulations of manifolds also in higher dimensions and codimensions, see [35], [36, Chap. 4]. It turned out that the tightness of a combinatorial triangulation is closely related to the concept of *Hamiltonicity* of a polyhedral complexes

(see [31, 36]): A subcomplex  $A$  of a polyhedral complex  $K$  is called  $k$ -Hamiltonian if  $A$  contains the full  $k$ -dimensional skeleton of  $K$  (not to be confused with the notion of a  $k$ -Hamiltonian graph [16]). This generalization of the notion of a Hamiltonian circuit in a graph seems to be due to Schulz [59, 60]. A Hamiltonian circuit then becomes a special case of a 0-Hamiltonian subcomplex of a 1-dimensional graph or of a higher-dimensional complex [24].

A triangulated  $2k$ -manifold that is a  $k$ -Hamiltonian subcomplex of the boundary complex of some higher dimensional simplex is a tight triangulation as Khnel [36, Chap. 4] showed. Such a triangulation is also called  $(k+1)$ -neighborly triangulation since any  $k+1$  vertices in a  $k$ -dimensional simplex are common neighbors. Moreover,  $(k+1)$ -neighborly triangulations of  $2k$ -manifolds are also referred to as *super-neighborly* triangulations – in analogy with neighborly polytopes the boundary complex of a  $(2k+1)$ -polytope can be at most  $k$ -neighborly unless it is a simplex. Notice here that combinatorial  $2k$ -manifolds can go beyond  $k$ -neighborliness, depending on their topology.

With the simplex as ambient polytope there exist generalized Heawood inequalities in even dimensions  $d \geq 4$  that were first conjectured by Khnel [35, 36], almost completely proved in [52] by Novik and proved by Novik and Swartz in [54]. As in the 2-dimensional case, the  $k$ -Hamiltonian triangulations of  $2k$ -manifolds here appear as regular cases of the generalized Heawood inequalities.

There also exist generalized Heawood inequalities for  $k$ -Hamiltonian subcomplexes of cross polytopes that were first conjectured by Sparla [62] and almost completely proved by Novik in [53]. The subcomplexes appearing as regular cases in these inequalities admit a tight embedding into a higher dimensional cross polytope and are also referred to as *nearly  $(k+1)$ -neighborly* as they contain all  $i$ -simplices,  $i \leq k$ , not containing one of the diagonals of the cross polytope (i.e. they are “neighborly except for the diagonals of the cross polytope”).

For  $d = 2$ , a regular case of Heawood’s inequality corresponds to a triangulation of an abstract surface that is tight.

Ringel [56] and Jungerman and Ringel [29] showed that distinct from the Klein bottle, the orientable surface of genus 2, and from the surface with  $\chi = -1$  (cf. [28]) all of the infinitely many regular cases of the Heawood inequality do occur and thus that there are infinitely many tight triangulations of surfaces.

In contrast, in dimensions  $d \geq 3$  there only exist a finite number of known examples of tight triangulations (see [40] for a census), apart from the trivial case of the boundary of a simplex and an infinite series of triangulations of sphere bundles over the circle due to Kühnel [36, 5B], [33].

Especially in odd dimensions it seems to be hard to give combinatorial conditions for the tightness of a triangulation and such conditions were not known so far. This work presents one such condition holding in any dimension  $d \geq 4$ .

In the course of proving the Lower Bound Conjecture (LBC) for 3- and 4-manifolds, D. Walkup [64] defined a class  $\mathcal{K}(d)$  of “certain especially simple” [64, p. 1] combinatorial manifolds as the set of all combinatorial  $d$ -manifolds that only have *stacked*  $(d-1)$ -spheres as vertex links as defined below:

**Definition 1.3 (stacked polytope, stacked sphere [64])**

- (i) A simplex is a stacked polytope and each polytope obtained from a stacked polytope by adding a pyramid over one of its facets is again stacked.
- (ii) A triangulation of the  $d$ -sphere  $S^d$  is called stacked  $d$ -sphere if it is combinatorially isomorphic to the boundary complex of a stacked  $(d+1)$ -polytope.

Thus, a stacked  $d$ -sphere can be understood as the combinatorial manifold obtained from the boundary of the  $(d+1)$ -simplex by successive stellar subdivisions of facets of the boundary complex  $\partial\Delta^{d+1}$  of the  $(d+1)$ -simplex (i.e. by successively subdividing facets of the complex  $K_i$ ,  $i = 0, 1, 2, \dots$ , by an inner vertex, where  $K_0 = \partial\Delta^{d+1}$ ). In this work we will give combinatorial conditions for the tightness of members of  $\mathcal{K}(d)$  holding in all dimensions  $d \geq 4$ . The main results of this paper are the following:

In Theorem 2.5 we show that any polar Morse function subject to a condition on the number of critical points of even and odd indices is a perfect function. This can be understood as a combinatorial analogon to Morse’s lacunary principle, see Remark 2.6.

This result is used in Theorem 3.2 in which it is shown that every 2-neighborly member of  $\mathcal{K}(d)$  is a tight triangulation for  $d \geq 4$ . Thus all *tight-neighborly* triangulations as defined in [45] are tight for  $d \geq 4$  (see Section 4).

The paper is organized as follows.

Section 2 begins with a short introduction to polyhedral Morse theory giving rise to a tightness definition of a triangulation in terms of (polyhedral) Morse theory followed by the investigation on a certain family of perfect Morse functions. The latter functions can be used to give a combinatorial condition for the tightness of odd-dimensional combinatorial manifolds in terms of properties of the vertex links of such manifolds.

In Section 3 the tightness of members of  $\mathcal{K}(d)$  is discussed, followed by a discussion of the tightness of tight-neighborly triangulations for  $d \geq 4$  in Section 4. Both sections include examples of triangulations for which the stated theorems hold.

In Section 5 the classes  $\mathcal{K}^k(d)$  of combinatorial manifolds are introduced as a generalization of Walkup’s class  $\mathcal{K}(d)$  and examples of manifolds in these classes are presented. Furthermore, an analogue of Walkup’s theorem [64, Thm. 5], [36, Prop. 7.2] for  $d = 6$  is proved, assuming the validity of the Generalized Lower Bound Conjecture 5.9. Finally, Section 6 focuses on subcomplexes of cross polytopes that lie in the class  $\mathcal{K}^k(d)$  for some  $k$ . Here, an example of a centrally symmetric triangulation of  $S^4 \times S^2 \in \mathcal{K}^2(6)$  as a 2-Hamiltonian subcomplex of the 8-dimensional cross polytope is given. This triangulation is part of a conjectured series of triangulations of sphere products as tight subcomplexes of cross polytopes.

## 2 Polar Morse functions and tightness

Apart from the homological definition given in Definition 1.1 and 1.2, tightness can also be defined in the language of Morse theory in a natural way: On one hand, the total absolute curvature of a smooth immersion  $X$  equals the average number of critical points of any nondegenerate height function on  $X$  in a suitable normalization. On the other hand, the Morse inequality shows that the normalized total absolute curvature of a compact smooth manifold  $M$  is bounded below by the rank of the total homology  $H_*(M)$  with respect to any field of coefficients, where tightness is equivalent to the case of equality in this bound, see [40].

As an extension to classical Morse theory (see [48] for an introduction to the field), Kühnel [34, 36] developed what one might refer to as a “polyhedral Morse theory”. Note that in this theory many, but not all concepts carry over from the smooth to the polyhedral case, see the survey articles [42] and [10] for a comparison of the two cases.

In the polyhedral case a discrete analogon to the Morse functions in classical Morse theory, are defined as follows:

**Definition 2.1 (rsl functions, [34, 36])** *Let  $M$  be a combinatorial manifold of dimension  $d$ . A function  $f : M \rightarrow \mathbb{R}$  is called regular simplexwise linear (rsl, for short), if  $f(v) \neq f(v')$  for any two vertices  $v \neq v'$  of  $M$  and  $f$  is linear when restricted to any simplex of  $M$ . Regular simplexwise linear functions are sometimes also referred to as Morse functions.*

Notice that an rsl function is uniquely determined by its value on the set of vertices and that only vertices can be critical points of  $f$  in the sense of Morse theory. With this definition at hand one can define critical points and levelsets of these Morse functions as in classical Morse theory:

**Definition 2.2 (critical vertices, [34, 36])** *Let  $\mathbb{F}$  be a field,  $M$  be a combinatorial  $d$ -manifold and let  $f$  be an rsl function on  $M$ . A vertex  $v \in M$  is called critical of index  $k$  and multiplicity  $m$  with respect to  $f$ , if*

$$\dim_{\mathbb{F}} H_k(M_v, M_v \setminus \{v\}; \mathbb{F}) = m > 0,$$

where  $M_v := \{x \in M : f(x) \leq f(v)\}$  and  $H_*$  denotes an appropriate homology theory with coefficients in  $\mathbb{F}$ . The number of critical points of  $f$  of index  $i$  (with multiplicity) are

$$\mu_i(f; \mathbb{F}) := \sum_{v \in V(M)} \dim_{\mathbb{F}} H_i(M_v, M_v \setminus \{v\}; \mathbb{F}).$$

In the following we will be interested in special kinds of Morse functions, so called *polar* Morse functions. This term was coined by Morse, see [51].

**Definition 2.3 (polar Morse function)** *Let  $f$  be a Morse function that only has one critical point of index 0 and of index  $d$  each for a given (necessarily connected)  $d$ -manifold. Then  $f$  is called polar Morse function.*

Note that for a 2-neighborly combinatorial manifold clearly all rsl functions are polar. As in the classical theory, there hold Morse relations as follows:

**Theorem 2.4 (Morse relations, [34, 36])** *Let  $\mathbb{F}$  be a field,  $M$  a combinatorial manifold of dimension  $d$  and  $f$  an rsl function on  $M$ . Then the following holds, where  $\beta_i(M; \mathbb{F}) := \dim_{\mathbb{F}} H_i(M; \mathbb{F})$  denotes the  $i$ -th Betti number:*

- (i)  $\mu_i(f; \mathbb{F}) \geq \beta_i(M; \mathbb{F})$  for all  $i$ ,
- (ii)  $\sum_{i=0}^d (-1)^i \mu_i(f; \mathbb{F}) = \chi(M) = \sum_{i=0}^d (-1)^i \beta_i(M; \mathbb{F})$ ,
- (iii)  $M$  is  $(k)$ -tight with respect to  $\mathbb{F}$  if and only if  $\mu_i(f; \mathbb{F}) = \beta_i(M; \mathbb{F})$  for every rsl function  $f$  and for all  $0 \leq i \leq d$  (for all  $0 \leq i \leq k$ ).

*Functions satisfying equality in (i) for all  $i \leq k$  are called  $k$ -tight functions. A function  $f$  that satisfies equality in (i) for all  $i$  is usually referred to as perfect or tight function, cf. [12].*

Note that a submanifold  $M$  of  $E^d$  is tight in the sense of Definition 1.1 if and only if every Morse function on  $M$  is a tight function, see [34, 36].

As already mentioned in Section 1, there exist quite a few examples of triangulations in even dimensions that are known to be tight, whereas “for odd-dimensional manifolds it seems to be difficult to transform the tightness of a polyhedral embedding into a simple combinatorial condition”, as Kühnel [36, Chap. 5] observed. Consequently there are few examples of triangulations of odd-dimensional manifolds that are known to be tight apart from the sporadic triangulations in [40] and Kühnel’s infinite series of  $S^{d-1} \times S^1$  for even  $d \geq 2$ .

It is a well known fact that in even dimensions a Morse function which only has critical points of even indices is a tight function, cf. [12]. This follows directly from the Morse relations, i.e. the fact that  $\sum_i (-1)^i \mu_i = \chi(M)$  holds for any Morse function on a manifold  $M$  and the fact that  $\mu_i \geq \beta_i$ . In odd dimensions on the other hand, arguing in this way is impossible as we always have  $\mu_0 \geq 1$  and the alternating sum allows the critical points to cancel out each other. What will be shown in Theorem 2.5 is that at least for a certain family of Morse functions the tightness of its members can readily be determined in arbitrary dimensions  $d \geq 3$ .

**Theorem 2.5** *Let  $\mathbb{F}$  be any field,  $d \geq 3$  and  $f$  a polar Morse function on a combinatorial  $\mathbb{F}$ -orientable  $d$ -manifold  $M$  such that the number of critical points of  $f$  (counted with multiplicity) satisfies*

$$\mu_{d-i}(f; \mathbb{F}) = \mu_i(f; \mathbb{F}) = \begin{cases} 0 & \text{for even } 2 \leq i \leq \lfloor \frac{d}{2} \rfloor \\ k_i & \text{for odd } 1 \leq i \leq \lfloor \frac{d}{2} \rfloor \end{cases},$$

*where  $k_i \geq 0$  for  $d$  arbitrary and moreover  $k_{\lfloor d/2 \rfloor} = k_{\lfloor d/2 \rfloor + 1} = 0$ , if  $d$  is odd. Then  $f$  is a tight function.*



*Proof.* Assume that the vertices  $v_1, \dots, v_n$  of  $M$  are ordered by their  $f$ -values,  $f(v_1) < f(v_2) < \dots < f(v_n)$ . In the long exact sequence for the relative homology

$$\begin{aligned} \dots \rightarrow H_{i+1}(M_v, M_v \setminus \{v\}) \rightarrow H_i(M_v \setminus \{v\}) \xrightarrow{\iota_i^*} H_i(M_v) \rightarrow \\ \rightarrow H_i(M_v, M_v \setminus \{v\}) \rightarrow H_{i-1}(M_v \setminus \{v\}) \rightarrow \dots \end{aligned} \quad (2.1)$$

the tightness of  $f$  is equivalent to the injectivity of the inclusion map  $\iota_i^*$  for all  $i$  and all  $v \in V(M)$ . The injectivity of  $\iota_i^*$  means that for any fixed  $j = 1, \dots, n$  the homology  $H_i(M_{v_j}, M_{v_{j-1}})$  (where  $M_{v_0} = \emptyset$ ) persists up to the maximal level  $H_i(M_{v_n}) = H_i(M)$  and is mapped injectively from level  $v_j$  to level  $v_{j+1}$ . This obviously is equivalent to the condition for tightness given in Definition 1.1. Thus tight triangulations can also be interpreted as triangulations with the maximal persistence of the homology in all dimensions with respect to the vertex ordering induced by  $f$  (see [21]). Hence, showing the tightness of  $f$  is equivalent to proving the injectivity of  $\iota_i^*$  at all vertices  $v \in V(M)$  and for all  $i$ , what will be done in the following.

The restriction of the number of critical points being nonzero only in every second dimension results in

$$\dim_{\mathbb{F}} H_i(M_v, M_v \setminus \{v\}) \leq \mu_i(f; \mathbb{F}) = 0$$

and

$$\dim_{\mathbb{F}} H_{d-i}(M_v, M_v \setminus \{v\}) \leq \mu_{d-i}(f; \mathbb{F}) = 0$$

and thus in  $H_i(M_v, M_v \setminus \{v\}) = H_{d-i}(M_v, M_v \setminus \{v\}) = 0$  for  $2 \leq i \leq \lfloor \frac{d}{2} \rfloor$  and all  $v \in V(M)$ , as  $M$  is  $\mathbb{F}$ -orientable. This implies a splitting of the long exact sequence (2.1) at every second dimension, yielding exact sequences of the forms

$$\begin{aligned} 0 \rightarrow H_{i-1}(M_v \setminus \{v\}) \xrightarrow{\iota_{i-1}^*} H_{i-1}(M_v) \rightarrow H_{i-2}(M_v, M_v \setminus \{v\}) \rightarrow \dots \quad \text{and} \\ 0 \rightarrow H_{d-i+1}(M_v \setminus \{v\}) \xrightarrow{\iota_{d-i+1}^*} H_{d-i+1}(M_v) \rightarrow H_{d-i+2}(M_v, M_v \setminus \{v\}) \rightarrow \dots, \end{aligned}$$

where the inclusions  $\iota_{i-1}^*$  and  $\iota_{d-i+1}^*$  are injective for all vertices  $v \in V(M)$ , again for  $2 \leq i \leq \lfloor \frac{d}{2} \rfloor$ . For critical points of index  $d$  the situation looks alike:

$$\begin{aligned} 0 \rightarrow H_d(M) \rightarrow H_d(M_v, M_v \setminus \{v\}) \rightarrow H_{d-1}(M_v \setminus \{v\}) \xrightarrow{\iota_{d-1}^*} H_{d-1}(M_v) \rightarrow \\ \rightarrow H_{d-2}(M_v, M_v \setminus \{v\}) \rightarrow \dots \end{aligned}$$

By assumption  $f$  only has one maximal vertex as it is polar. Then, if  $v$  is not the maximal vertex with respect to  $f$ ,  $H_d(M_v, M_v \setminus \{v\}) = 0$  and thus  $\iota_{d-1}^*$  is injective. If, on the other hand,  $v$  is the maximal vertex with respect to  $f$ , one has

$$H_d(M) \cong H_d(M_v, M_v \setminus \{v\}),$$

as  $M_v \setminus \{v\} = M \setminus \{v\}$  in this case. Consequently, by the exactness of the sequence above,  $\iota_{d-1}^*$  is also injective in this case. Altogether it follows that  $\iota_i^*$  is injective for all  $i$  and for all vertices  $v \in V(M)$  and thus that  $f$  is  $\mathbb{F}$ -tight.  $\square$

As we will see in Section 3, this is a condition that can be translated into a purely combinatorial one. Examples of manifolds to which Theorem 2.5 applies will be given in the following sections.

### Remark 2.6

- (i) Theorem 2.5 can be understood as a combinatorial equivalent of Morse's *lacunary principle* [13, Lecture 2]. The lacunary principle in the smooth case states that if  $f$  is a smooth Morse function on a smooth manifold  $M$  such that its Morse polynomial  $M_t(f)$  contains no consecutive powers of  $t$ , then  $f$  is a perfect Morse function.

- (ii) Due to the Morse relations, Theorem 2.5 puts a restriction on the topology of manifolds admitting these kinds of Morse functions. In particular, these must have vanishing Betti numbers in the dimensions where the number of critical points is zero. Note that in dimension  $d = 3$  the theorem thus only holds for homology 3-spheres with  $\beta_1 = \beta_2 = 0$  and no statements concerning the tightness of triangulations with  $\beta_1 > 0$  can be made. One way of proving the tightness of a 2-neighborly combinatorial 3-manifold  $M$  would be to show that the mapping

$$H_2(M_v) \rightarrow H_2(M, M_v \setminus \{v\}) \quad (2.2)$$

is surjective for all  $v \in V(M)$ . This would result in an injective mapping in the homology group  $H_1(M_v \setminus \{v\}) \rightarrow H_1(M_v)$  for all  $v \in V(M)$  – as above by virtue of the long exact sequence for the relative homology – and thus in the 1-tightness of  $M$ , which is equivalent to the  $(\mathbb{F}_2)$ -tightness of  $M$  for  $d = 3$ , see [36, Prop. 3.18]. Unfortunately, there does not seem to be an easy to check combinatorial condition on  $M$  that is sufficient for the surjectivity of the mapping (2.2), in contrast to the case of a combinatorial condition for the 0-tightness of  $M$  for which this is just the 2-neighborliness of  $M$ .

### 3 Tightness of members of $\mathcal{K}(d)$

In this section we will investigate the tightness of members of Walkup's class  $\mathcal{K}(d)$ , the family of all combinatorial  $d$ -manifolds that only have stacked  $(d - 1)$ -spheres as vertex links. For  $d \leq 2$ ,  $\mathcal{K}(d)$  is the set of all triangulated  $d$ -manifolds. Kalai [30] showed that the stacking-condition of the links puts a rather strong topological restriction on the members of  $\mathcal{K}(d)$ :

**Theorem 3.1 (Kalai, [30, 6])** *Let  $d \geq 4$ . Then  $M$  is a connected member of  $\mathcal{K}(d)$  if and only if  $M$  is obtained from a stacked  $d$ -sphere by  $\beta_1(M)$  combinatorial handle additions.*

Here a *combinatorial handle addition* to a complex  $C$  is defined as usual (see [64, 30, 45]) as the complex  $C^\psi$  obtained from  $C$  by identifying two facets  $\Delta_1$  and  $\Delta_2$  of  $C$  such that  $v \in V(\Delta_1)$  is identified with  $w \in \Delta_2$  only if  $d(v, w) \geq 3$ , where  $V(X)$  denotes the vertex set of a simplex  $X$  and  $d(v, w)$  the distance of the vertices  $v$  and  $w$  in the 1-skeleton of  $C$  seen as undirected graph (cf. [5]).

In other words Kalai's theorem states that any connected  $M \in \mathcal{K}(d)$  is necessarily homeomorphic to a connected sum with summands of the form  $S^1 \times S^{d-1}$  and  $S^1 \times S^{d-1}$ , compare [45]. Looking at 2-neighborly members of  $\mathcal{K}(d)$ , the following observation concerning the embedding of the triangulation can be made:

**Theorem 3.2** *Let  $d = 2$  or  $d \geq 4$ . Then any 2-neighborly member of  $\mathcal{K}(d)$  yields a tight triangulation of the underlying PL manifold.*

Note that since any triangulated 1-sphere is stacked,  $\mathcal{K}(2)$  is the set of all triangulated surfaces and that any 2-neighborly triangulation of a surface is tight. The two conditions of the manifold being 2-neighborly and having only stacked spheres as vertex links are rather strong as the only stacked sphere that is  $k$ -neighborly,  $k \geq 2$ , is the boundary of the simplex, see also Remark 5.5. Thus the only  $k$ -neighborly member of  $\mathcal{K}(d)$ ,  $k \geq 3$ ,  $d \geq 2$ , is the boundary of the  $(d + 1)$ -simplex.

The following lemma will be needed for the proof of Theorem 3.2.

**Lemma 3.3** *Let  $S$  be a stacked  $d$ -sphere,  $d \geq 3$ , and  $V' \subseteq V(S)$ . Then  $H_{d-j}(\text{span}_S(V')) = 0$  for  $2 \leq j \leq d - 1$ , where  $H_\star$  denotes the simplicial homology groups.*

*Proof.* Assume that  $S_0 = \partial \Delta^{d+1}$  and assume  $S_{i+1}$  to be obtained from  $S_i$  by a single stacking operation such that there exists an  $N \in \mathbb{N}$  with  $S_N = S$ . Then  $S_{i+1}$  is obtained from  $S_i$  by removing a facet of  $S_i$  and the boundary of a new  $d$ -simplex  $T_i$  followed by a gluing operation of  $S_i$  and  $T_i$  along the boundaries of the removed facets. This process can also be understood in

terms of a bistellar 0-move carried out on a facet of  $S_i$ . Since this process does not remove any  $(d-1)$ -simplices from  $S_i$  or  $T_i$  we have  $\text{skel}_{d-1}(S_i) \subset \text{skel}_{d-1}(S_{i+1})$ .

We prove the statement by induction on  $i$ . Clearly, the statement is true for  $i = 0$ , as  $S_0 = \partial\Delta^{d+1}$  and  $\partial\Delta^{d+1}$  is  $(d+1)$ -neighborly. Now assume that the statement holds for  $S_i$  and let  $V'_{i+1} \subset V(S_{i+1})$ . In the following we can consider the connected components  $C_k$  of  $\text{span}_{S_{i+1}}(V'_{i+1})$  separately. If  $C_k \subset S_i$  or  $C_k \subset T_i$  then the statement is true by assumption and the  $(d+1)$ -neighborliness of  $\partial\Delta^{d+1}$ , respectively. Otherwise let  $P_1 := C_k \cap S_i \neq \emptyset$  and  $P_2 := C_k \cap T_i \neq \emptyset$ . Then

$$H_{d-j}(P_1) \cong H_{d-j}(P_1 \cap T_i) \text{ and } H_{d-j}(P_2) \cong H_{d-j}(P_2 \cap S_i).$$

This yields

$$\begin{aligned} H_{d-j}(P_1 \cup P_2) &= H_{d-j}((P_1 \cup P_2) \cap S_i \cap T_i) \\ &= H_{d-j}(\text{span}_{S_i \cap T_i}(V'_{i+1})) \\ &= H_{d-j}(\text{span}_{S_i \cap T_i}(V'_{i+1} \cap V(S_i \cap T_i))) \\ &= 0, \end{aligned}$$

as  $S_i \cap T_i = \partial\Delta^d$  which is  $(d-1)$ -neighborly such that the span of any vertex set has vanishing  $(d-j)$ -th homology for  $2 \leq j \leq d-1$ .  $\square$

*Proof (of Theorem 3.2).* For  $d = 2$ , see [36] for a proof. From now on assume that  $d \geq 4$ . As can be shown via excision (see, for example [34]), if  $M$  is a combinatorial  $d$ -manifold,  $f : M \rightarrow \mathbb{R}$  an rsl function on  $M$  and  $v \in V(M)$ , then

$$H_*(M_v, M_v \setminus \{v\}) \cong H_*(M_v \cap \text{st}(v), M_v \cap \text{lk}(v)).$$

Now let  $d \geq 4$ ,  $1 < i < d-1$ . The long exact sequence for the relative homology

$$\begin{aligned} \dots &\rightarrow H_{d-i}(M_v \cap \text{st}(v)) \rightarrow H_{d-i}(M_v \cap \text{st}(v), M_v \cap \text{lk}(v)) \rightarrow \\ &\rightarrow H_{d-i-1}(M_v \cap \text{lk}(v)) \rightarrow H_{d-i-1}(M_v \cap \text{st}(v)) \rightarrow \dots \end{aligned}$$

yields an isomorphism

$$H_{d-i}(M_v \cap \text{st}(v), M_v \cap \text{lk}(v)) \cong H_{d-i-1}(M_v \cap \text{lk}(v)), \quad (3.1)$$

as  $H_{d-i}(M_v \cap \text{st}(v)) = H_{d-i-1}(M_v \cap \text{st}(v)) = 0$ . Note here, that  $M_v \cap \text{st}(v)$  is a cone over  $M_v \cap \text{lk}(v)$  and thus contractible.

Since  $M \in \mathcal{K}(d)$ , all vertex links in  $M$  are stacked  $(d-1)$ -spheres and thus Lemma 3.3 applies to the right hand side of (3.1). This implies that a  $d$ -manifold  $M \in \mathcal{K}(d)$ ,  $d \geq 4$ , cannot have critical points of index  $2 \leq i \leq d-2$ , i.e.  $\mu_2(f; \mathbb{F}) = \dots = \mu_{d-2}(f; \mathbb{F}) = 0$ .

Furthermore, the 2-neighborliness of  $M$  implies that any function on  $M$  is polar. Thus all prerequisites of Theorem 2.5 are fulfilled,  $f$  is tight and consequently  $M$  is a tight triangulation, what was to be shown.  $\square$

**Remark 3.4** In even dimensions  $d \geq 4$ , Theorem 3.2 can also be proved without using Theorem 2.5. In this case the statement follows from the 2-neighborliness of  $M$  (that yields  $\mu_0(f; \mathbb{F}) = \beta_0$  and  $\mu_d(f; \mathbb{F}) = \beta_d$ ), and the Morse relations 2.4 which then yield  $\mu_1(f; \mathbb{F}) = \beta_1$  and  $\mu_{d-1}(f; \mathbb{F}) = \beta_{d-1}$  for any rsl function  $f$ , as  $\mu_2(f; \mathbb{F}) = \dots = \mu_{d-2}(f; \mathbb{F}) = 0$ .

As a consequence, the stacking condition of the links already implies the vanishing of  $\beta_2, \dots, \beta_{d-2}$  (as by the Morse relations  $\mu_i \geq \beta_i$ ), in accordance with Kalai's Theorem 3.1.

An example of a series of tight combinatorial manifolds is the infinite series of sphere bundles over the circle due to Kühnel [33]. The triangulations in this series are all 2-neighborly on  $f_0 = 2d+3$  vertices. They are homeomorphic to  $S^{d-1} \times S^1$  in even dimensions and to  $S^{d-1} \times S^1$  in odd dimensions. Furthermore, all links are stacked and thus Theorem 3.2 applies providing an alternative proof of the tightness of the triangulations in this series:

**Corollary 3.5** *All members  $M^d$  of the series of triangulations in [33] are 2-neighborly and lie in the class  $\mathcal{K}(d)$ . They are thus tight triangulations by Theorem 3.2.*

Another example of a triangulation to which Theorem 3.2 applies is due to Bagchi and Datta [6] and will be given in the following section. It is an example of a so called *tight-neighborly* triangulation as defined by Lutz, Sulanke and Swartz [45]. For this class of manifolds, Theorem 3.2 holds for  $d = 2$  and  $d \geq 4$ . Tight-neighborly triangulations will be described in more detail in the next section.

## 4 Tight-neighborly triangulations

Beside the class of combinatorial  $d$ -manifolds with stacked spheres as vertex links  $\mathcal{K}(d)$ , Walkup [64] also defined the class  $\mathcal{H}(d)$ . This is the family of all simplicial complexes that can be obtained from the boundary complex of the  $(d+1)$ -simplex by a series of zero or more of the following three operations: (i) stellar subdivision of facets, (ii) combinatorial handle additions and (iii) forming connected sums of objects obtained from the first two operations.

The two classes are closely related. Obviously, the relation  $\mathcal{H}(d) \subset \mathcal{K}(d)$  holds. Kalai [30] showed the reverse inclusion  $\mathcal{K}(d) \subset \mathcal{H}(d)$  for  $d \geq 4$ .

Note that the condition of the 2-neighborliness of an  $M \in \mathcal{K}(d)$  in Theorem 3.2 is equivalent to the first Betti number  $\beta_1(M)$  being maximal with respect to the vertex number  $f_0(M)$  of  $M$  (as a 2-neighborly triangulation does not allow any handle additions). Such manifolds are exactly the cases of equality in [54, Th. 5.2]. In their recent work [45], Lutz, Sulanke and Swartz prove the following<sup>1</sup>

**Theorem 4.1 (Theorem 5 in [45])** *Let  $\mathbb{K}$  be any field and let  $M$  be a  $\mathbb{K}$ -orientable triangulated  $d$ -manifold with  $d \geq 3$ . Then*

$$f_0(M) \geq \left\lceil \frac{1}{2} \left( 2d + 3 + \sqrt{1 + 4(d+1)(d+2)\beta_1(M; \mathbb{K})} \right) \right\rceil. \quad (4.1)$$

**Remark 4.2** As pointed out in [45], for  $d = 2$  inequality (4.1) coincides with Heawood's inequality

$$f_0(M) \geq \left\lceil \frac{1}{2} \left( 7 + \sqrt{49 - 24\chi(M)} \right) \right\rceil$$

if one replaces  $\beta_1(M; \mathbb{K})$  by  $\frac{1}{2}\beta_1(M; \mathbb{K})$  to account for the double counting of the middle Betti number  $\beta_1(M; \mathbb{K})$  of surfaces by Poincaré duality. Inequality (4.1) can also be written in the form

$$\binom{f_0 - d - 1}{2} \geq \binom{d+2}{2} \beta_1.$$

Thus Theorem 5 in [45] settles Kühnel's conjectured bounds

$$\binom{f_0 - d + j - 2}{j+1} \geq \binom{d+2}{j+1} \beta_j \quad \text{with } 1 \leq j \leq \lfloor \frac{d-1}{2} \rfloor$$

in the case  $j = 1$ .

For  $\beta_1 = 1$ , the bound (4.1) coincides with the Brehm-Kühnel bound [14]  $f_0 \geq 2d + 4 - j$  for  $(j-1)$ - but not  $j$ -connected  $d$ -manifolds in the case  $j = 1$ . Inequality (4.1) is sharp by the series of vertex minimal triangulations of sphere bundles over the circle presented in [33].

Triangulations of connected sums of sphere bundles  $(S^2 \times S^1)^{\#k}$  and  $(S^2 \times S^1)^{\#k}$  attaining equality in (4.1) for  $d = 3$  were discussed in [45]. Such triangulations are necessarily 2-neighborly and Lutz, Sulanke and Swartz defined:

<sup>1</sup>The author would like to thank Frank Lutz for fruitful discussions about tight-neighborly triangulations and pointing him to the work [45] in the first place.

$\beta_1$	$d$	$n$	type	reference
0	any $d$	$d + 1$	$S^{d-1}$	$\partial\Delta^d$
1	any even $d \geq 2$	$2d + 3$	$S^{d-1} \times S^1$	$M^d$ in [33]
1	any odd $d \geq 2$	$2d + 3$	$S^{d-1} \times S^1$	$M^d$ in [33]
1	2	7	$S^1 \times S^1$	Möbius' 7 vertex torus [50, 19], $M^2$ in [33]
1	3	9	$S^2 \times S^1$	$H_-^3$ in [64], $N_{51}^9$ in [3], $M^3$ in [33]
2	13	35	?	
3	4	15	$(S^3 \times S^1)^{\#3}$	$M_{15}^4$ in [6]
5	5	21	?	
8	10	44	?	

Table 1: Known and open cases for  $\beta_1$ ,  $d$  and  $n$  of two-neighborly members of  $\mathcal{K}(d)$ .

**Definition 4.3 (tight-neighborly triangulation, [45])** *Let  $d \geq 2$  and let  $M$  be a triangulation of  $(S^{d-1} \times S^1)^{\#k}$  or  $(S^{d-1} \times S^1)^{\#k}$  attaining equality in (4.1). Then  $M$  is called a tight-neighborly triangulation.*

For  $d \geq 4$ , all triangulations of  $\mathbb{F}$ -orientable  $\mathbb{F}$ -homology  $d$ -manifolds with equality in (4.1) lie in  $\mathcal{H}(d)$  and are tight-neighborly triangulations of  $(S^{d-1} \times S^1)^{\#k}$  or  $(S^{d-1} \times S^1)^{\#k}$  by Theorem 5.2 in [54].

The authors conjectured [45, Conj. 13] that all tight-neighborly triangulations are tight in the classical sense of Definition 1.2 and showed that the conjecture holds in the following cases: for  $\beta_1 = 0, 1$  and any  $d$  and for  $d = 2$  and any  $\beta_1$ . Indeed, the conjecture also holds for any  $d \geq 4$  and any  $\beta_1$  as it is a direct consequence of Theorem 3.2:

**Corollary 4.4** *For  $d \geq 4$  all tight-neighborly triangulations are tight.*

*Proof.* For  $d \geq 4$  one has  $\mathcal{H}(d) = \mathcal{K}(d)$  and the statement is true for all 2-neighborly members of  $\mathcal{K}(d)$  by Theorem 3.2.  $\square$

It remains to be investigated whether for vertex minimal triangulations of  $d$ -handlebodies,  $d \geq 3$ , the reverse implication is true, too, i. e. that for this class of triangulations the terms of tightness and tight-neighborliness are equivalent.

**Question 4.5** *Let  $d \geq 4$  and let  $M$  be a tight triangulation homeomorphic to  $(S^{d-1} \times S^1)^{\#k}$  or  $(S^{d-1} \times S^1)^{\#k}$ . Does this imply that  $M$  is tight-neighborly?*

As was shown in [45], at least for values of  $\beta_1 = 0, 1$  and any  $d$  and for  $d = 2$  and any  $\beta_1$  this is true.

One example of a handlebody for which Theorem 3.2 holds is due to Bagchi and Datta [6]. The triangulation  $M_{15}^4$  of  $(S^3 \times S^1)^{\#3}$  from [6] is a 2-neighborly combinatorial 4-manifold on 15 vertices that is a member of  $\mathcal{K}(4)$  with  $f$ -vector  $f = (15, 105, 230, 240, 96)$ . Since  $M_{15}^4$  is tight-neighborly, we have the following

**Corollary 4.6** *The 4-manifold  $M_{15}^4$  given in [6] is a tight triangulation.*

The next possible triples of values of  $\beta_1$ ,  $d$  and  $n$  for which a 2-neighborly member of  $\mathcal{K}(d)$  could exist (compare [45]) are listed in Table 1. Apart from the sporadic examples in dimension 4 and the infinite series for of higher dimensional analogues of Császár's torus in arbitrary dimension  $d \geq 2$  due to Khnel [33], cf. [39, 5, 18], mentioned earlier, no further examples are known as of today.

Especially in (the odd) dimension  $d = 3$  things seem to be a bit more subtle, as already laid out in Remark 2.6. As Perles [2] showed that the link of any vertex in a neighborly 4-polytope is stacked (compare also Remark 5 in [30]), we know that the class  $\mathcal{K}(3)$  is rather big compared to

$\mathcal{H}(3)$ . Thus a statement equivalent to Theorem 3.2 is not surprisingly false for members of  $\mathcal{K}(3)$ , a counterexample being the boundary of the cyclic polytope  $\partial C(4, 6)$  which is 2-neighborly but certainly not a tight triangulation as it has empty triangles. The only currently known example of a tight-neighborly combinatorial 3-manifold is a 9-vertex triangulation of  $S^2 \times S^1$  independently found by Walkup [64] and Altshuler and Steinberg [3]. This triangulation is combinatorially unique, as was shown by Bagchi and Datta [7]. For  $d = 3$  it is open whether there exist tight-neighborly triangulations for higher values of  $\beta_1 \geq 2$ , see [45, Question 12].

This leaves a 9-vertex triangulation  $M^3$  of  $S^2 \times S^1$  due to Walkup [64] as the only currently known 3-dimensional tight-neighborly triangulation. The fact that  $M^3$  is a tight triangulation is well known, see [36]. Yet, we will present here another proof of the tightness of  $M^3$ . It is a rather easy procedure when looking at the 4-polytope  $P$  the boundary of which  $M^3$  was constructed from by one elementary combinatorial handle addition, see also [6].

**Lemma 4.7** *Walkup's 9-vertex triangulation  $M^3$  of  $S^2 \times S^1$  is tight.*

*Proof.* Take the stacked 4-polytope  $P$  with  $f$ -vector  $f(P) = (13, 42, 58, 37, 9)$  from [64]. Its facets are

$$\begin{array}{lll} \langle 12345 \rangle, & \langle 23456 \rangle, & \langle 34567 \rangle, \\ \langle 45678 \rangle, & \langle 56789 \rangle, & \langle 678910 \rangle, \\ \langle 7891011 \rangle, & \langle 89101112 \rangle, & \langle 910111213 \rangle. \end{array}$$

As  $P$  is stacked it has missing edges (called *diagonals*), but no empty faces of higher dimension.

Take the boundary  $\partial P$  of  $P$ . By construction,  $P$  has no inner  $i$ -faces,  $i \leq 2$  so that  $\partial P$  has the 36 diagonals of  $P$  and additionally 8 empty tetrahedra, but no empty triangles. As  $\partial P$  is a 3-sphere, the empty tetrahedra are all homologous to zero.

Now form a 1-handle over  $\partial P$  by removing the two tetrahedra  $[1, 2, 3, 4]$  and  $[10, 11, 12, 13]$  from  $\partial P$  followed by an identification of the four vertex pairs  $(i, i + 9)$ ,  $1 \leq i \leq 4$ , where the newly identified vertices are labeled with  $1, \dots, 4$ .

This process yields a 2-neighborly combinatorial manifold  $M^3$  with  $13 - 4 = 9$  vertices and one additional empty tetrahedron  $[1, 2, 3, 4]$ , which is the generator of  $H_2(M)$ .

As  $M^3$  is 2-neighborly it is 0-tight and as  $\partial P$  had no empty triangles, two empty triangles in the span of any vertex subset  $V' \subset V(M)$  are always homologous. Thus  $M^3$  is a tight triangulation.  $\square$

The construction in the proof above could probably be used in the general case with  $d = 3$  and  $\beta_1 \geq 2$ : one starts with a stacked 3-sphere  $M_0$  as the boundary of a stacked 4-polytope which by construction does not contain empty 2-faces and then successively forms handles over this boundary 3-sphere (obtaining triangulated manifolds  $M_1, \dots, M_n = M$ ) until the resulting triangulation  $M$  is 2-neighborly and fulfills equality in (4.1). Note that this can only be done in the regular cases of (4.1), i.e. where (4.1) admits integer solutions for the case of equality. See [45] for a list of possible configurations.

## 5 $k$ -stacked spheres and the class $\mathcal{K}^k(d)$

McMullen and Walkup [47] extended the notion of stacked polytopes to  $k$ -stacked polytopes as simplicial  $d$ -polytopes that can be triangulated without introducing new  $j$ -faces for  $0 \leq j \leq d - k - 1$ . More generally, we can define:

**Definition 5.1 ( $k$ -stacked balls and spheres, [47, 30])** *A  $k$ -stacked  $(d + 1)$ -ball,  $0 \leq k \leq d$ , is a triangulated  $(d + 1)$ -ball that has no interior  $j$ -faces,  $0 \leq j \leq d - k$ . A minimally  $k$ -stacked  $(d + 1)$ -ball is a  $k$ -stacked  $(d + 1)$ -ball that is not  $(k - 1)$ -stacked. The boundary of any (minimally)  $k$ -stacked  $(d + 1)$ -ball is called a (minimally)  $k$ -stacked  $d$ -sphere.*

Note that in this context the ordinary stacked  $d$ -spheres are exactly the 1-stacked  $d$ -spheres. Note also that a  $k$ -stacked  $d$ -sphere is obviously also  $(k + l)$ -stacked, where  $l \in \mathbb{N}$ ,  $k + l \leq d$ , compare

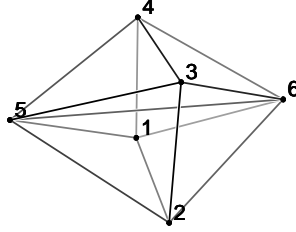


Figure 1: A minimally 2-stacked  $S^2$  as the boundary complex of a subdivided 3-octahedron.

[4]. The simplex  $\Delta^{d+1}$  is the only 0-stacked  $(d+1)$ -ball and the boundary of the simplex  $\partial\Delta^{d+1}$  is the only 0-stacked  $d$ -sphere. Keep in mind that all triangulated  $d$ -spheres are at least  $d$ -stacked [4, Rem. 9.1].

Figure 1 shows the boundary of an octahedron as an example of a minimally 2-stacked 2-sphere  $S$  with 6 vertices. The octahedron that is subdivided along the inner diagonal  $(5,6)$  can be regarded as a triangulated 3-ball  $B$  with  $\text{skel}_0(S) = \text{skel}_0(B)$  and  $\partial B = S$ . Note that although all vertices of  $B$  are on the boundary, there is an inner edge such that the boundary is 2-stacked, but not 1-stacked. In higher dimensions, examples of minimally  $d$ -stacked  $d$ -spheres exist as boundary complexes of subdivided  $d$ -cross polytopes with an inner diagonal.

Akin to the 1-stacked case, a more geometrical characterization of  $k$ -stacked  $d$ -spheres can be given via bistellar moves (also known as Pachner moves, see [55]), at least for  $k \leq \lfloor \frac{d}{2} \rfloor$ :

**Definition 5.2 (bistellar moves)** *Let  $M$  be a triangulated  $d$ -manifold and let  $A$  be a  $(d-i)$ -face of  $M$ ,  $0 \leq i \leq d$ , such that there exists an  $i$ -simplex  $B$  that is not a face of  $M$  with  $\text{lk}_M(A) = \partial B$ . Then a bistellar  $i$ -move  $\Phi_A$  on  $M$  is defined by*

$$\Phi_A(M) := (M \setminus (A * \partial B)) \cup (\partial A * B),$$

where  $*$  denotes the join operation for simplicial complexes. Bistellar  $i$ -moves with  $i > \lfloor \frac{d}{2} \rfloor$  are also-called reverse  $(d-i)$ -moves.

See Figure 2 for an example illustration of bistellar moves in dimension  $d = 3$ . Note that for any bistellar move  $\Phi_A(M)$ ,  $A * B$  forms a  $(d+1)$ -simplex. Thus any sequence of bistellar moves defines a sequence of  $(d+1)$ -simplices – this we will call the *induced sequence of  $(d+1)$ -simplices* in the following.

The characterization of  $k$ -stacked  $d$ -spheres using bistellar moves is the following. For  $k \leq \lfloor \frac{d}{2} \rfloor$ , a complex  $S$  obtained from the boundary of the  $(d+1)$ -simplex by a sequence of bistellar  $i$ -moves,  $0 \leq i \leq k-1$ , is a  $k$ -stacked  $d$ -sphere: As  $k \leq \lfloor \frac{d}{2} \rfloor$ , the sequence of  $(d+1)$ -simplices induced by the sequence of bistellar moves is duplicate free and defines a simplicial  $(d+1)$ -ball  $B$  with  $\partial B = S$ . Furthermore,  $\text{skel}_{d-k}(B) = \text{skel}_{d-k}(S)$  holds as no bistellar move in the sequence can contribute an inner  $j$ -face to  $B$ ,  $0 \leq j \leq d-k$ . Thus  $S$  is a  $k$ -stacked  $d$ -sphere. As the above arguments also apply to reverse  $i$ -moves,  $0 \leq i \leq k-1$ , vice versa, any  $k$ -stacked  $d$ -sphere with  $k \leq \lfloor \frac{d}{2} \rfloor$  is bistellarly equivalent to the boundary of the  $d$ -simplex by only applying reverse  $i$ -moves,  $0 \leq i \leq k-1$ . Thus any  $k$ -stacked sphere with  $k \leq \lfloor \frac{d}{2} \rfloor$  is combinatorial.

Keep in mind though, that this interpretation does not hold for values  $k > \lfloor \frac{d}{2} \rfloor$  as in this case the sequence of  $(d+1)$ -simplices induced by the sequence of bistellar moves may have duplicate entries, as opposed to the case with  $k \leq \lfloor \frac{d}{2} \rfloor$ .

In terms of bistellar moves, the minimally 2-stacked sphere in Figure 1 can be constructed as follows: Start with a solid tetrahedron and stack another tetrahedron onto one of its facets. Now introduce the inner diagonal  $(5,6)$  via a 1-move. Clearly, this complex is not bistellarly equivalent to the simplex by only applying reverse 0-moves (and thus not (1-)stacked) but it is bistellarly equivalent to the simplex by only applying reverse 0-, and 1-moves and thus minimally 2-stacked.

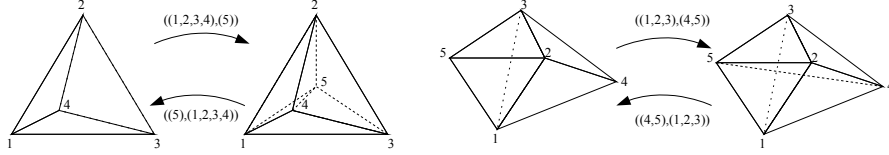


Figure 2: Left: a 0-move on a tetrahedron and its inverse 3-move (i. e. a reverse 0-move). Right: a 1-move on two tetrahedra glued together at one triangle and its inverse 2-move (i. e. a reverse 1-move).

The author is one of the authors of the toolkit `simpcomp` [23] for simplicial constructions in the `GAP` system [25]. `simpcomp` contains an algorithm that checks whether a given  $d$ -sphere is  $k$ -stacked,  $k \leq \lfloor \frac{d}{2} \rfloor$ , using the argument above.

With the notion of  $k$ -stacked spheres at hand one can define:

**Definition 5.3 (the class  $\mathcal{K}^k(d)$ )** Let  $\mathcal{K}^k(d)$ ,  $k \leq d$ , be the family of all  $d$ -dimensional simplicial complexes all whose vertex links are  $k$ -stacked spheres.

Note that  $\mathcal{K}^d(d)$  is the set of all triangulated manifolds for any  $d$  and that Walkup's class  $\mathcal{K}(d)$  coincides with  $\mathcal{K}^1(d)$  above. In analogy to the 1-stacked case, a  $(k+1)$ -neighborly member of  $\mathcal{K}^k(d)$  with  $d \geq 2k$  necessarily has vanishing  $\beta_1, \dots, \beta_{k-1}$ . Thus it seems reasonable to ask for the existence of a generalization of Kalai's Theorem 3.1 to the class of  $\mathcal{K}^k(d)$  for  $k \geq 2$ .

Furthermore, one might be tempted to ask for a generalization of Theorem 3.2 to the class  $\mathcal{K}^k(d)$  for  $k \geq 2$ . Unfortunately, there seems to be no direct way of generalizing Theorem 3.2 to also hold for members of  $\mathcal{K}^k(d)$  giving a combinatorial condition for the tightness of such triangulations. The key obstruction here is the fact that a generalization of Lemma 3.3 is impossible. While in the case of ordinary stacked spheres a bistellar 0-move does not introduce inner simplices to the  $(d-1)$ -skeleton, the key argument in Lemma 3.3, this is not true for bistellar  $i$ -moves for  $i \geq 1$ .

Nonetheless an analogous result to Theorem 3.2 should be true for such triangulations:

**Question 5.4** Let  $d \geq 4$  and  $2 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$  and let  $M$  be a  $(k+1)$ -neighborly combinatorial manifold such that  $M \in \mathcal{K}^k(d)$ . Does this imply the tightness of  $M$ ?

**Remark 5.5** Note that all vertex links of  $(k+1)$ -neighborly members of  $\mathcal{K}^k(d)$  are thus  $k$ -stacked  $k$ -neighborly spheres and that McMullen and Walkup [47, Sect. 3] showed that there exist  $k$ -stacked  $k$ -neighborly spheres on  $n$  vertices for any  $2 \leq 2k \leq d < n$ . Some examples of such spheres will be given in the following. The conditions of being  $k$ -stacked and  $k$ -neighborly at the same time is strong as the two conditions tend to exclude each other in the following sense: McMullen and Walkup showed that if a  $d$ -sphere is  $k$ -stacked and  $k'$ -neighborly with  $k' > k$ , then it is the boundary of the simplex. In that sense the  $k$ -stacked  $k$ -neighborly spheres appear as the most strongly restricted non-trivial objects of this class, where a stronger restriction only admits the simplex as the trivial solution. Thus the conditions in Theorem 3.2 (with  $k = 1$ ) and in Question 5.4 are the most restrictive ones still admitting non-trivial solutions: If one asks that the links are minimally  $l$ -stacked with  $l < k$  instead of minimally  $k$ -stacked or if one demands the complexes to be  $(k+m)$ -neighborly,  $m > 1$ , instead of just  $(k+1)$ -neighborly, this only leaves the boundary complex of the simplex as a possible solution.

Kühnel and Lutz [40] gave an overview of the currently known tight triangulations. The statement of Question 5.4 holds for all the triangulations listed in [40] – note although that there even exist  $k$ -neighborly triangulations in  $\mathcal{K}^k(d)$  that are tight and thus fail to fulfill the prerequisites of Question 5.4 (see Table 2).



dim	top. type	$n$	nb.	$k$
4	$\mathbb{C}P^2$	9	3	2
4	$K3$	16	3	2
4	$(S^3 \times S^1) \# (\mathbb{C}P^2)^{\#5}$	15	2	2
5	$S^3 \times S^2$	12	3	2
5	$SU(3)/SO(3)$	13	3	3
6	$S^3 \times S^3$	13	4	3

Table 2: Some known tight triangulations and their membership in the classes  $\mathcal{K}^k(d)$ , cf. [40], with  $n$  denoting the number of vertices of the triangulation and  $nb.$  its neighborliness.

Although we did not succeed in proving conditions for the tightness of triangulations lying in  $\mathcal{K}^k(d)$ ,  $k \geq 2$ , these have nonetheless interesting properties that we will investigate upon in the following. Also, many known tight triangulations are members of these classes, as will be shown. Our first observation is that the neighborliness of a triangulation is closely related to the property of being a member of  $\mathcal{K}^k(d)$ :

**Lemma 5.6** *Let  $k \in \mathbb{N}$  and  $M$  be a combinatorial  $d$ -manifold,  $d \geq 2k$ , that is a  $(k+1)$ -neighborly triangulation. Then  $M \in \mathcal{K}^{d-k}(d)$ .*

*Proof.* If  $M$  is  $(k+1)$ -neighborly, then for any  $v \in V(M)$ ,  $\text{lk}(v)$  is  $k$ -neighborly. As  $\text{lk}(v)$  is PL homeomorphic to  $\partial\Delta^d$  (since  $M$  is a combinatorial manifold) there exists a  $d$ -ball  $B$  with  $\partial B = \text{lk}(v)$  (cf. [4]). Since  $\text{lk}(v)$  is  $k$ -neighborly,  $\text{skel}_{k-1}(B) = \text{skel}_{k-1}(\text{lk}(v))$ . By Definition 5.1 the link of every vertex  $v \in V(M)$  then is  $(d-k)$ -stacked and thus  $M \in \mathcal{K}^{d-k}(d)$ .  $\square$

As pointed out in Section 1, Kühnel [36, Chap. 4] investigated  $(k+1)$ -neighborly triangulations of  $2k$ -manifolds and showed that all these are tight triangulations. By Lemma 5.6 all their vertex links are  $k$ -stacked spheres:

**Corollary 5.7** *Let  $M$  be a  $(k+1)$ -neighborly (tight) triangulation of a  $2k$ -manifold. Then  $M$  lies in  $\mathcal{K}^k(2k)$ .*

In particular this holds for many vertex minimal (tight) triangulations of 4-manifolds:

**Corollary 5.8** *The known examples of the vertex-minimal tight triangulation of a  $K3$ -surface with  $f$ -vector  $f = (16, 120, 560, 720, 288)$  due to Casella and Kühnel [15] and the unique vertex-minimal tight triangulation of  $\mathbb{C}P^2$  with  $f$ -vector  $f = (9, 36, 84, 90, 36)$  due to Kühnel [38], cf. [37] are 3-neighborly triangulations that lie in  $\mathcal{K}^2(4)$ .*

Let us now shed some light on properties of members of  $\mathcal{K}^2(6)$ . First recall that there exists a *Generalized Lower Bound Conjecture* (GLBC) due to McMullen and Walkup as an extension to the classical Lower Bound Theorem for triangulated spheres as follows:

**Conjecture 5.9 (GLBC, cf. [47, 4])** *For  $d \geq 2k+1$ , the face-vector  $(f_0, \dots, f_d)$  of any triangulated  $d$ -sphere  $S$  satisfies*

$$f_j \geq \begin{cases} \sum_{i=-1}^{k-1} (-1)^{k-i+1} \binom{j-i-1}{j-k} \binom{d-i+1}{j-i} f_i, & \text{if } k \leq j \leq d-k, \\ \sum_{i=-1}^{k-1} (-1)^{k-i+1} \left[ \binom{j-i-1}{j-k} \binom{d-i+1}{j-i} - \binom{k}{d-j+1} \binom{d-i}{d-k+1} \right] f_i, & \text{if } d-k+1 \leq j \leq d. \end{cases} \quad (5.1)$$

*Equality holds here for some  $j$  if and only if  $S$  is a  $k$ -stacked  $d$ -sphere.*

The GLBC implies the following theorem for  $d = 6$ , which is a 6-dimensional analogue of Walkup's theorem [64, Thm. 5], [36, Prop. 7.2], see also Swartz' Theorem 4.10 in [63]:

**Theorem 5.10** *Assuming the validity of the Generalized Lower Bound Conjecture 5.9, for any combinatorial 6-manifold  $M$  the inequality*

$$f_2(M) \geq 28\chi(M) - 21f_0 + 6f_1 \quad (5.2)$$

*holds. If  $M$  is 2-neighborly, then*

$$f_2(M) \geq 28\chi(M) + 3f_0(f_0 - 8) \quad (5.3)$$

*holds. In either case equality is attained if and only if  $M \in \mathcal{K}^2(6)$ .*

*Proof.* Clearly,

$$f_3(M) = \frac{1}{4} \sum_{v \in V(M)} f_2(\text{lk}(v)). \quad (5.4)$$

By applying the GLBC 5.9 to all the vertex links of  $M$  one obtains a lower bound on  $f_2(\text{lk}(v))$  for all  $v \in V(M)$ :

$$f_2(\text{lk}(v)) \geq 35 - 15f_0(\text{lk}(v)) + 5f_1(\text{lk}(v)). \quad (5.5)$$

Here equality is attained if and only if  $\text{lk}(v)$  is 2-stacked. Combining (5.4) and (5.5) yields a lower bound

$$\begin{aligned} f_3(M) &\geq \frac{1}{4} \sum_{v \in V(M)} 35 - 15f_0(\text{lk}(v)) + 5f_1(\text{lk}(v)) \\ &= \frac{5}{4} (7f_0(M) - 6f_1(M) + 3f_2(M)), \end{aligned} \quad (5.6)$$

for which equality holds if and only if  $M \in \mathcal{K}^2(6)$ .

If we eliminate  $f_4, \dots, f_6$  from the Dehn-Sommerville-Equations for combinatorial 6-manifolds, we obtain the linear equation

$$35f_0 - 15f_1 + 5f_2 - f_3 = 35\chi(M). \quad (5.7)$$

Inserting inequality (5.6) into (5.7) and solving for  $f_2(M)$  yields the claimed lower bounds (5.2) and (5.3),

$$\begin{aligned} f_2(M) &\geq 28\chi(M) - 21f_0(M) + 6f_1(M) \\ &= 28\chi(M) + 3f_0(M) - \underbrace{f_0(M) - 8}_{\geq 0}, \end{aligned} \quad (5.8)$$

where the 2-neighborliness of  $M$  was used in the last line.  $\square$

For a possible 14-vertex triangulation of  $S^4 \times S^2$  (with  $\chi = 4$ ) inequality (5.8) becomes

$$f_2 \geq 4 \cdot 28 + 3 \cdot 14 \cdot (14 - 8) = 364,$$

but together with the trivial upper bound  $f_2 \leq \binom{f_0}{3}$  this already would imply that such a triangulation necessarily is 3-neighborly, as  $\binom{14}{3} = 364$ .

So, just by asking for a 2-neighborly combinatorial  $S^4 \times S^2$  on 14 vertices that lies in  $\mathcal{K}^2(6)$  already implies that this triangulation is 3-neighborly. Also, the example would attain equality in the Brehm-Khnel bound [14] as an example of a 1-connected 6-manifold with 14 vertices. We strongly conjecture that this triangulation is tight, see Question 5.4.

## 6 Subcomplexes of the cross polytope

The  $d$ -dimensional cross polytope (or  $d$ -octahedron)  $\beta^d$  is defined as the convex hull of the  $2d$  points

$$(0, \dots, 0, \pm 1, 0, \dots, 0) \in \mathbb{R}^d.$$

It is a simplicial and regular polytope and it is centrally-symmetric with  $d$  missing edges called *diagonals*, each between two antipodal points of type  $(0, \dots, 0, 1, 0, \dots, 0)$  and  $(0, \dots, 0, -1, 0, \dots, 0)$ . Its edge graph is the complete  $d$ -partite graph with two vertices in each partition, sometimes denoted by  $K_2 * \dots * K_2$ . See [46, 26, 65] for properties of regular polytopes in general.

The boundary of the  $(d+1)$ -cross polytope  $\beta^{d+1}$  is an obviously minimally  $d$ -stacked  $d$ -sphere as it can be obtained as the boundary of a minimally  $d$ -stacked  $(d+1)$ -ball that is given by any subdivision of  $\beta^{d+1}$  along an inner diagonal.

As pointed out in Section 1, centrally symmetric analogues of tight triangulations appear as Hamiltonian subcomplexes of cross polytopes. A *centrally symmetric triangulation* is a triangulation such that there exists a combinatorial involution operating on the face lattice of the triangulation without fixed points. Any centrally symmetric triangulation thus has an even number of vertices and can be interpreted as a subcomplex of some higher dimensional cross polytope. The tightness of a centrally symmetric  $(k-1)$ -connected  $2k$ -manifold  $M$  as a subcomplex of  $\beta^d$  then is equivalent to  $M$  being a  $k$ -Hamiltonian subcomplex of  $\beta^d$ , i.e. that  $M$  is nearly  $(k+1)$ -neighborly, see [36, Ch. 4].

As it turns out, all of the known centrally symmetric triangulations of  $d$ -manifolds that are  $k$ -Hamiltonian subcomplexes of a higher dimensional cross polytope  $\beta^N$  and admit a tight embedding into  $\beta^N$  lie in the class  $\mathcal{K}^k(d)$ . This will be discussed in the following paragraphs.

**Corollary 6.1** *The 16-vertex triangulation of  $(S^2 \times S^2)_{16}^{\#7}$  presented in [22] lies in  $\mathcal{K}^2(4)$  and admits a tight embedding into  $\beta^8$  as shown in [22].*

*Proof.* The triangulation  $(S^2 \times S^2)_{16}^{\#7}$  is a combinatorial manifold and a tight subcomplex of  $\beta^8$  as shown in [22]. Thus each vertex link is a PL 3-sphere. It remains to show that all vertex links are 2-stacked.

The 2-stackedness of the links is equivalent to the condition that all links are bistellarly equivalent to the boundary of the 4-simplex by employing only reverse bistellar 0- and 1-moves to the vertex links. As mentioned earlier, an algorithm for this is implemented in `simpcomp`. The algorithm in [23] gave a positive answer for all links, such that  $(S^2 \times S^2)_{16}^{\#7} \in \mathcal{K}^2(4)$ , as claimed.

The following centrally symmetric triangulation of  $S^4 \times S^2$  is a new example of a triangulation that can be seen as a subcomplex of a higher dimensional cross polytope.

**Theorem 6.2** *There exists an example of a centrally symmetric triangulation  $M_{16}^6$  of  $S^4 \times S^2$  with 16 vertices that is a 2-Hamiltonian subcomplex of the 8-octahedron  $\beta^8$  and that lies in  $\mathcal{K}^2(6)$ .*

*Proof.* The construction of  $M_{16}^6$  was done entirely with `simpcomp` [23] and is as follows. First a 24-vertex triangulation  $\tilde{M}^6$  of  $S^4 \times S^2$  was constructed as the standard simplicial cartesian product of  $\partial\Delta^3$  and  $\partial\Delta^5$  as implemented in [23], where  $\Delta^d$  denotes the  $d$ -simplex. Then  $\tilde{M}^6$  obviously is a combinatorial 6-manifold homeomorphic to  $S^4 \times S^2$ .

This triangulation  $\tilde{M}^6$  was then reduced to the triangulation  $M_{16}^6$  with  $f$ -vector  $f = (16, 112, 448, 980, 1232, 840, 240)$  using a vertex reduction algorithm based on bistellar flips that is implemented in [23]. The code is based on the vertex reduction methods developed by Björner and Lutz [11]. It is well-known that this reduction process leaves the PL type of the triangulation invariant such that  $M_{16}^6 \cong S^4 \times S^2$ . The 240 5-simplices of  $M_{16}^6$  are given in Table 3. The  $f$ -vector of  $M_{16}^6$  is uniquely determined already by the condition of  $M_{16}^6$  to be 2-Hamiltonian in the 8-dimensional cross polytope. In particular,  $M_{16}^6$  has 8 missing edges of the form  $\langle i, i+1 \rangle$  for all odd  $1 \leq i \leq 15$ , which are pairwise disjoint and correspond to the 8 diagonals of the cross polytope. As there is an involution

$$I = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)$$

operating on the faces of  $M_{16}^6$  without fixed points,  $M^2$  can be seen as a 2-Hamiltonian subcomplex of  $\beta^8$ .

It remains to show that  $M_{16}^6 \in \mathcal{K}^2(6)$ . Remember that the necessary and sufficient condition for a triangulation  $X$  to lie in  $\mathcal{K}^k(d)$  is that all vertex links of  $X$  are  $k$ -stacked  $(d-1)$ -spheres. Since  $M_{16}^6$  is a combinatorial 6-manifold, all vertex links are triangulated 5-spheres. It thus suffices to

show that all vertex links are 2-stacked. As above, the 2-stackedness of the links is equivalent to the condition that all links are bistellarly equivalent to the boundary of the 5-simplex by employing only bistellar reverse 0- and reverse 1-moves to the vertex links. `simpcomp` gave a positive answer for all links, such that  $M_{16}^6 \in \mathcal{K}^2(6)$  follows, as claimed.  $\square$

The triangulation  $M_{16}^6$  is strongly conjectured to be tight in  $\beta^8$ . It is part of a conjectured series of centrally symmetric triangulations of sphere products for which tight embeddings into cross polytopes are conjectured. In some cases, the tightness of the embedding could be proved, see [62], [40, 6.2] and [22, Sect. 6]. In particular the sphere products presented in [40, Thm. 6.3] are part of this conjectured series and the following holds:

**Theorem 6.3** *The centrally symmetric triangulations of sphere products of the form  $S^k \times S^m$  with vertex transitive automorphism groups*

$$\begin{array}{cccccccc} S^1 \times S^1, & S^2 \times S^1, & S^3 \times S^1, & S^4 \times S^1, & S^5 \times S^1, & S^6 \times S^1, & S^7 \times S^1, & \\ & & S^2 \times S^2, & S^3 \times S^2, & & S^5 \times S^2, & & \\ & & & & S^3 \times S^3, & S^4 \times S^3, & S^5 \times S^3, & \\ & & & & & & S^4 \times S^4, & \end{array}$$

on  $n = 2(k + m) + 4$  vertices presented in [40, Thm. 6.3] all lie in the class  $\mathcal{K}^{\min\{k,m\}}(k + m)$ .

This can be easily shown by using `simpcomp` [23] to verify that the vertex links of these triangulations are minimally  $\min\{k, m\}$ -stacked spheres. Since these examples all have a transitive automorphism group, it suffices to check the stackedness condition for one vertex link only.

The preceding observations naturally lead to the following Question 6.4 as a generalization of Question 5.4:

**Question 6.4** Let  $d \geq 4$  and let  $M$  be a  $k$ -Hamiltonian codimension 2 subcomplex of the  $(d + 2)$ -dimensional cross polytope  $\beta^{d+2}$  such that  $M \in \mathcal{K}^k(d)$  for some fixed  $1 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$ . Does this imply that the embedding  $M \subset \beta^{d+2} \subset E^{d+2}$  is tight?

This is true for all currently known codimension 2 subcomplexes of cross polytopes that admit a tight embedding into the polytope: The 8-vertex triangulation of the torus, a 12-vertex triangulation of  $S^2 \times S^2$  due to Sparla [43, 61] and the triangulations of  $S^k \times S^k$  on  $4k + 4$  vertices for  $k = 3$  and  $k = 4$  as well as for the infinite series of triangulations of  $S^k \times S^1$  in [33]. For the other triangulations of  $S^k \times S^m$  listed in Theorem 6.3 above, Khnel and Lutz “strongly conjecture” [40, Sec. 6] that they are tight in the  $(k + m + 2)$ -dimensional cross polytope. Nevertheless it is currently not clear whether the conditions of Question 6.4 imply the tightness of the embedding into the cross polytope.

In accordance with [40, Conjecture 6.2] we then have the following

**Conjecture 6.5** *Any centrally symmetric combinatorial triangulation  $M_n^{k+m}$  of  $S^k \times S^m$  on  $n = 2(k + m) + 4$  vertices is tight if regarded as a subcomplex of the  $\frac{n}{2}$ -dimensional cross polytope.  $M_n^{k+m}$  lies in the class  $\mathcal{K}^{\min\{k,m\}}(k + m)$ .*

## Acknowledgement

The author acknowledges support by the Deutsche Forschungsgemeinschaft (DFG). This work was carried out as part of the DFG project Ku 1203/5-2.

## References

- [1] Aleksandr D. Alexandrov. On a class of closed surfaces. *Recueil Math. (Moscow)*, 4:69–72, 1938.
- [2] Amos Altshuler and Leon Steinberg. Neighborly 4-polytopes with 9 vertices. *J. Combinatorial Theory Ser. A*, 15:270–287, 1973.

Table 3: The 240 5-simplices of  $M_{16}^6$ .

(1 2 3 4 7 12 14),	(1 2 3 4 7 12 16),	(1 2 3 4 7 13 14),	(1 2 3 4 7 13 16),	(1 2 3 4 9 12 14),
(1 2 3 4 9 12 16),	(1 2 3 4 9 14 16),	(1 2 3 4 13 14 16),	(1 2 3 6 7 12 14),	(1 2 3 6 7 12 16),
(1 2 3 6 7 13 14),	(1 2 3 6 7 13 16),	(1 2 3 6 9 10 12),	(1 2 3 6 9 10 13),	(1 2 3 6 9 12 16),
(1 2 3 6 9 13 16),	(1 2 3 6 10 11 12),	(1 2 3 6 10 11 13),	(1 2 3 6 11 12 14),	(1 2 3 6 11 13 14),
(1 2 3 9 10 11 12),	(1 2 3 9 10 11 13),	(1 2 3 9 11 12 14),	(1 2 3 9 11 13 14),	(1 2 3 9 13 14 16),
(1 2 4 7 12 14 15),	(1 2 4 7 12 15 16),	(1 2 4 7 13 14 15),	(1 2 4 7 13 15 16),	(1 2 4 9 12 14 16),
(1 2 4 12 14 15 16),	(1 2 4 13 14 15 16),	(1 2 6 7 12 14 16),	(1 2 6 7 13 14 15),	(1 2 6 7 13 15 16),
(1 2 6 7 14 15 16),	(1 2 6 9 10 11 12),	(1 2 6 9 10 11 13),	(1 2 6 9 11 12 14),	(1 2 6 9 11 13 15),
(1 2 6 9 11 14 15),	(1 2 6 9 12 14 16),	(1 2 6 9 13 15 16),	(1 2 6 9 14 15 16),	(1 2 6 11 13 14 15),
(1 2 7 12 14 15 16),	(1 2 9 11 13 14 15),	(1 2 9 13 14 15 16),	(1 3 4 7 12 14 16),	(1 3 4 7 13 14 16),
(1 3 4 9 12 14 16),	(1 3 6 7 12 14 16),	(1 3 6 7 13 14 16),	(1 3 6 8 9 10 11),	(1 3 6 8 9 10 13),
(1 3 6 8 9 11 14),	(1 3 6 8 9 13 14),	(1 3 6 8 10 11 13),	(1 3 6 8 11 13 14),	(1 3 6 9 10 11 12),
(1 3 6 9 11 12 14),	(1 3 6 9 12 14 16),	(1 3 6 9 13 14 16),	(1 3 8 9 10 11 13),	(1 3 8 9 11 13 14),
(1 4 7 8 10 11 13),	(1 4 7 8 10 11 15),	(1 4 7 8 10 13 16),	(1 4 7 8 10 15 16),	(1 4 7 8 11 13 15),
(1 4 7 8 12 14 15),	(1 4 7 8 12 14 16),	(1 4 7 8 12 15 16),	(1 4 7 8 13 14 15),	(1 4 7 8 13 14 16),
(1 4 7 10 11 13 15),	(1 4 7 10 13 15 16),	(1 4 8 10 11 13 15),	(1 4 8 10 13 15 16),	(1 4 8 12 14 15 16),
(1 4 8 13 14 15 16),	(1 6 7 8 10 11 13),	(1 6 7 8 10 11 15),	(1 6 7 8 10 13 16),	(1 6 7 8 10 15 16),
(1 6 7 8 11 13 15),	(1 6 7 8 13 14 15),	(1 6 7 8 13 14 16),	(1 6 7 8 14 15 16),	(1 6 7 10 11 13 15),
(1 6 7 10 13 15 16),	(1 6 8 9 10 11 15),	(1 6 8 9 10 13 16),	(1 6 8 9 10 15 16),	(1 6 8 9 11 14 15),
(1 6 8 9 13 14 16),	(1 6 8 9 14 15 16),	(1 6 8 11 13 14 15),	(1 6 9 10 11 13 15),	(1 6 9 10 13 15 16),
(1 7 8 12 14 15 16),	(1 8 9 10 11 13 15),	(1 8 9 10 13 15 16),	(1 8 9 11 13 14 15),	(1 8 9 13 14 15 16),
(2 3 4 5 7 10 11),	(2 3 4 5 7 10 16),	(2 3 4 5 7 11 14),	(2 3 4 5 7 14 16),	(2 3 4 5 9 10 11),
(2 3 4 5 9 10 12),	(2 3 4 5 9 11 14),	(2 3 4 5 9 12 16),	(2 3 4 5 9 14 16),	(2 3 4 5 10 12 16),
(2 3 4 7 10 11 12),	(2 3 4 7 10 12 16),	(2 3 4 7 11 12 14),	(2 3 4 7 13 14 16),	(2 3 4 9 10 11 12),
(2 3 4 9 11 12 14),	(2 3 5 6 9 10 12),	(2 3 5 6 9 10 13),	(2 3 5 6 9 11 13),	(2 3 5 6 9 11 14),
(2 3 5 6 9 12 16),	(2 3 5 6 9 14 16),	(2 3 5 6 10 11 12),	(2 3 5 6 10 11 13),	(2 3 5 6 10 12 14),
(2 3 5 6 12 14 16),	(2 3 5 7 10 11 12),	(2 3 5 7 10 12 16),	(2 3 5 7 11 12 14),	(2 3 5 7 12 14 16),
(2 3 5 9 10 11 13),	(2 3 6 7 12 14 16),	(2 3 6 7 13 14 16),	(2 3 6 9 11 13 14),	(2 3 6 9 13 14 16),
(2 4 5 7 10 11 12),	(2 4 5 7 10 12 15),	(2 4 5 7 10 15 16),	(2 4 5 7 11 12 14),	(2 4 5 7 12 14 15),
(2 4 5 7 14 15 16),	(2 4 5 9 10 11 12),	(2 4 5 9 11 12 14),	(2 4 5 9 12 14 16),	(2 4 5 10 12 15 16),
(2 4 5 12 14 15 16),	(2 4 7 10 12 15 16),	(2 4 7 13 14 15 16),	(2 5 6 9 10 11 12),	(2 5 6 9 10 11 13),
(2 5 6 9 11 12 14),	(2 5 6 9 12 14 16),	(2 5 7 10 12 15 16),	(2 5 7 12 14 15 16),	(2 6 7 13 14 15 16),
(2 6 9 11 13 14 15),	(2 6 9 13 14 15 16),	(3 4 5 7 8 10 11),	(3 4 5 7 8 10 16),	(3 4 5 7 8 11 12),
(3 4 5 7 8 12 16),	(3 4 5 7 11 12 14),	(3 4 5 7 12 14 16),	(3 4 5 8 9 10 11),	(3 4 5 8 9 10 12),
(3 4 5 8 9 11 12),	(3 4 5 8 10 12 16),	(3 4 5 9 11 12 14),	(3 4 5 9 12 14 16),	(3 4 7 8 10 11 12),
(3 4 7 8 10 12 16),	(3 4 8 9 10 11 12),	(3 5 6 8 9 10 12),	(3 5 6 8 9 10 13),	(3 5 6 8 9 11 12),
(3 5 6 8 9 11 13),	(3 5 6 8 10 11 12),	(3 5 6 8 10 11 13),	(3 5 6 9 11 12 14),	(3 5 6 9 12 14 16),
(3 5 7 8 10 11 12),	(3 5 7 8 10 12 16),	(3 5 8 9 10 11 13),	(3 6 8 9 10 11 12),	(3 6 8 9 11 13 14),
(4 5 7 8 10 11 13),	(4 5 7 8 10 13 16),	(4 5 7 8 11 12 15),	(4 5 7 8 11 13 15),	(4 5 7 8 12 14 15),
(4 5 7 8 12 14 16),	(4 5 7 8 13 14 15),	(4 5 7 8 13 14 16),	(4 5 7 10 11 12 15),	(4 5 7 10 11 13 15),
(4 5 7 10 13 15 16),	(4 5 7 13 14 15 16),	(4 5 8 9 10 11 13),	(4 5 8 9 10 12 15),	(4 5 8 9 10 13 15),
(4 5 8 9 11 12 15),	(4 5 8 9 11 13 15),	(4 5 8 10 12 15 16),	(4 5 8 10 13 15 16),	(4 5 8 12 14 15 16),
(4 5 8 13 14 15 16),	(4 5 9 10 11 12 15),	(4 5 9 10 11 13 15),	(4 7 8 10 11 12 15),	(4 7 8 10 12 15 16),
(4 8 9 10 11 12 15),	(4 8 9 10 11 13 15),	(5 6 7 8 10 11 13),	(5 6 7 8 10 11 15),	(5 6 7 8 10 13 15),
(5 6 7 8 11 13 15),	(5 6 7 10 11 13 15),	(5 6 8 9 10 12 15),	(5 6 8 9 10 13 15),	(5 6 8 9 11 12 15),
(5 6 8 9 11 13 15),	(5 6 8 10 11 12 15),	(5 6 9 10 11 12 15),	(5 6 9 10 11 13 15),	(5 7 8 10 11 12 15),
(5 7 8 10 12 15 16),	(5 7 8 10 13 15 16),	(5 7 8 12 14 15 16),	(5 7 8 13 14 15 16),	(6 7 8 10 13 15 16),
(6 7 8 13 14 15 16),	(6 8 9 10 11 12 15),	(6 8 9 10 13 15 16),	(6 8 9 11 13 14 15),	(6 8 9 13 14 15 16).

- [3] Amos Altshuler and Leon Steinberg. Neighborly combinatorial 3-manifolds with 9 vertices. *Discrete Math.*, 8:113–137, 1974.
- [4] Bhaskar Bagchi and Basudeb Datta. Lower bound theorem for normal pseudomanifolds. *Expo. Math.*, 26(4):327–351, 2008.
- [5] Bhaskar Bagchi and Basudeb Datta. Minimal triangulations of sphere bundles over the circle. *J. Combin. Theory Ser. A*, 115(5):737–752, 2008.
- [6] Bhaskar Bagchi and Basudeb Datta. On Walkup’s class  $\mathcal{K}(d)$  and a minimal triangulation of a 4-manifold. [arXiv:0804.2153v1 \[math.GT\]](#), Preprint, 8 pages, 2008.
- [7] Bhaskar Bagchi and Basudeb Datta. Uniqueness of Walkup’s 9-vertex 3-dimensional Klein bottle. *Discrete Math.*, 308(22):5087–5095, 2008.
- [8] Thomas F. Banchoff. Tightly embedded 2-dimensional polyhedral manifolds. *Amer. J. Math.*, 87:462–472, 1965.
- [9] Thomas F. Banchoff. Tight polyhedral Klein bottles, projective planes, and Möbius bands. *Math. Ann.*, 207:233–243, 1974.
- [10] Thomas F. Banchoff and Wolfgang Kühnel. Tight submanifolds, smooth and polyhedral. In *Tight and taut submanifolds (Berkeley, CA, 1994)*, volume 32 of *Math. Sci. Res. Inst. Publ.*, pages 51–118. Cambridge Univ. Press, Cambridge, 1997.
- [11] Anders Björner and Frank H. Lutz. Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere. *Experiment. Math.*, 9(2):275–289, 2000.
- [12] Raoul Bott. Morse theory and the Yang-Mills equations. In *Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979)*, volume 836 of *Lecture Notes in Math.*, pages 269–275. Springer, Berlin, 1980.
- [13] Raoul Bott. Lectures on Morse theory, old and new. *Bull. Amer. Math. Soc. (N.S.)*, 7(2):331–358, 1982.
- [14] Ulrich Brehm and Wolfgang Kühnel. Combinatorial manifolds with few vertices. *Topology*, 26(4):465–473, 1987.
- [15] Mario Casella and Wolfgang Kühnel. A triangulated  $K3$  surface with the minimum number of vertices. *Topology*, 40(4):753–772, 2001.
- [16] Gary Chartrand and S. F. Kapoor. The cube of every connected graph is 1-hamiltonian. *J. Res. Nat. Bur. Standards Sect. B*, 73B:47–48, 1969.

- [17] Shiing-Shen Chern and Richard K. Lashof. On the total curvature of immersed manifolds. *Amer. J. Math.*, 79:306–318, 1957.
- [18] Jacob Chestnut, Jenya Sapir, and Ed Swartz. Enumerative properties of triangulations of spherical bundles over  $S^1$ . *European J. Combin.*, 29(3):662–671, 2008.
- [19] Ákos Császár. A polyhedron without diagonals. *Acta Univ. Szeged. Sect. Sci. Math.*, 13:140–142, 1949.
- [20] Basudeb Datta. Minimal triangulations of manifolds. *J. Indian Inst. Sci.*, 87(4):429–449, 2007.
- [21] Herbert Edelsbrunner and John Harer. Persistent homology—a survey. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 257–282. Amer. Math. Soc., Providence, RI, 2008.
- [22] Felix Effenberger and Wolfgang Kühnel. Hamiltonian submanifolds of regular polytopes. *Discrete Comput. Geom.*, 43(2):242–262, March 2010. Preprint available: [arXiv:0709.3998v2](https://arxiv.org/abs/0709.3998v2) [math.CO].
- [23] Felix Effenberger and Jonathan Spreer. simpcomp - A GAP package, Version 1.3.0. <http://www.igt.uni-stuttgart.de/LstDiffgeo/simpcomp>, 2010. Submitted to the GAP Group.
- [24] Günter Ewald. Hamiltonian circuits in simplicial complexes. *Geometriae Dedicata*, 2:115–125, 1973.
- [25] GAP – Groups, Algorithms, and Programming, Version 4.4.12. <http://www.gap-system.org>, 2008.
- [26] Branko Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.
- [27] P. J. Heawood. Map colour theorem. *Quart. J. Math.*, 24:332–338, 1890.
- [28] John Philip Huneke. A minimum-vertex triangulation. *J. Combin. Theory Ser. B*, 24(3):258–266, 1978.
- [29] M. Jungerman and G. Ringel. Minimal triangulations on orientable surfaces. *Acta Math.*, 145(1-2):121–154, 1980.
- [30] Gil Kalai. Rigidity and the lower bound theorem. I. *Invent. Math.*, 88(1):125–151, 1987.
- [31] W. Kühnel. Hamiltonian surfaces in polytopes. In *Intuitive geometry (Szeged, 1991)*, volume 63 of *Colloq. Math. Soc. János Bolyai*, pages 197–203. North-Holland, Amsterdam, 1994.
- [32] Wolfgang Kühnel. Tight and 0-tight polyhedral embeddings of surfaces. *Invent. Math.*, 58(2):161–177, 1980.
- [33] Wolfgang Kühnel. Higher dimensional analogues of Császár’s torus. *Results Math.*, 9:95–106, 1986.
- [34] Wolfgang Kühnel. Triangulations of manifolds with few vertices. In *Advances in differential geometry and topology*, pages 59–114. World Sci. Publ., Teaneck, NJ, 1990.
- [35] Wolfgang Kühnel. Manifolds in the skeletons of convex polytopes, tightness, and generalized Heawood inequalities. In *Polytopes: abstract, convex and computational (Scarborough, ON, 1993)*, volume 440 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 241–247. Kluwer Acad. Publ., Dordrecht, 1994.
- [36] Wolfgang Kühnel. *Tight polyhedral submanifolds and tight triangulations*, volume 1612 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.
- [37] Wolfgang Kühnel and Thomas F. Banchoff. The 9-vertex complex projective plane. *Math. Intelligencer*, 5(3):11–22, 1983.
- [38] Wolfgang Kühnel and Gunter Lassmann. The unique 3-neighborly 4-manifold with few vertices. *J. Combin. Theory Ser. A*, 35(2):173–184, 1983.
- [39] Wolfgang Kühnel and Gunter Lassmann. Permuted difference cycles and triangulated sphere bundles. *Discrete Math.*, 162(1-3):215–227, 1996.
- [40] Wolfgang Kühnel and Frank H. Lutz. A census of tight triangulations. *Period. Math. Hungar.*, 39(1-3):161–183, 1999. Discrete geometry and rigidity (Budapest, 1999).
- [41] Nicolaas H. Kuiper. Immersions with minimal total absolute curvature. In *Colloque Géom. Diff. Globale (Bruxelles, 1958)*, pages 75–88. Centre Belge Rech. Math., Louvain, 1959.
- [42] Nicolaas H. Kuiper. Geometry in total absolute curvature theory. In *Perspectives in mathematics*, pages 377–392. Birkhäuser, Basel, 1984.
- [43] Gunter Lassmann and Eric Sparla. A classification of centrally-symmetric and cyclic 12-vertex triangulations of  $S^2 \times S^2$ . *Discrete Math.*, 223(1-3):175–187, 2000.
- [44] Frank H. Lutz. Triangulated Manifolds with Few Vertices: Combinatorial Manifolds. [arXiv:math/0506372v1](https://arxiv.org/abs/math/0506372v1) [math.CO], Preprint, 37 pages, 2005.
- [45] Frank H. Lutz, Thom Sulanke, and Ed Swartz.  $f$ -vectors of 3-manifolds. *Electron. J. Comb.*, 16(2):Research paper R13, 33 p., 2009.
- [46] Peter McMullen and Egon Schulte. *Abstract regular polytopes*, volume 92 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.
- [47] Peter McMullen and David W. Walkup. A generalized lower bound conjecture for simplicial polytopes. *Mathematika*, 18:264–273, 1971.
- [48] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

- [49] John Milnor. On the relationship between the Betti numbers of a hypersurface and an integral of its Gaussian curvature (1950). In *Collected papers. Vol. 1, Geometry*, pages 15 – 26. Publish or Perish Inc., Houston, TX, 1994.
- [50] August Möbius. *Gesammelte Werke*, Vol. 2. Verlag Hirzel, Leipzig, 1886.
- [51] Marston Morse. The existence of polar non-degenerate functions on differentiable manifolds. *Ann. of Math. (2)*, 71:352–383, 1960.
- [52] Isabella Novik. Upper bound theorems for homology manifolds. *Israel J. Math.*, 108:45–82, 1998.
- [53] Isabella Novik. On face numbers of manifolds with symmetry. *Adv. Math.*, 192(1):183–208, 2005.
- [54] Isabella Novik and Ed Swartz. Socles of Buchsbaum modules, complexes and posets. *Advances in Mathematics*, 222(6):2059–2084, 2009.
- [55] Udo Pachner. Konstruktionsmethoden und das kombinatorische Homöomorphieproblem für Triangulierungen kompakter semilinearer Mannigfaltigkeiten. *Abh. Math. Sem. Uni. Hamburg*, 57:69–86, 1987.
- [56] Gerhard Ringel. Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlegen kann. *Math. Ann.*, 130:317–326, 1955.
- [57] Gerhard Ringel. *Map color theorem*. Springer-Verlag, New York, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 209.
- [58] Colin P. Rourke and Brian J. Sanderson. *Introduction to piecewise-linear topology*. Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69.
- [59] Christoph Schulz. *Mannigfaltigkeiten mit Zellzerlegung im Randkomplex eines konvexen Polytops und verallgemeinerte Hamilton-Kreise*. Dissertation, Universität Bochum, 1974.
- [60] Christoph Schulz. Polyhedral manifolds on polytopes. *Rend. Circ. Mat. Palermo (2) Suppl.*, (35):291–298, 1994. First International Conference on Stochastic Geometry, Convex Bodies and Empirical Measures (Palermo, 1993).
- [61] Eric Sparla. *Geometrische und kombinatorische Eigenschaften triangulierter Mannigfaltigkeiten*. Berichte aus der Mathematik. [Reports from Mathematics]. Verlag Shaker, Aachen, 1997. Dissertation, Universität Stuttgart.
- [62] Eric Sparla. A new lower bound theorem for combinatorial  $2k$ -manifolds. *Graphs Combin.*, 15(1):109–125, 1999.
- [63] Ed Swartz. Face enumeration - from spheres to manifolds. *J. Europ. Math. Soc.*, 11:449–485, 2009.
- [64] David W. Walkup. The lower bound conjecture for 3- and 4-manifolds. *Acta Math.*, 125:75–107, 1970.
- [65] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

Felix Effenberger  
 Pfaffenwaldring 57  
 70569 Stuttgart  
 Germany

**E-Mail:** effenberger@mathematik.uni-stuttgart.de

**WWW:** <http://www.igt.uni-stuttgart.de/LstDiffgeo/Effenberger/>





## Erschienenene Preprints ab Nummer 2007/001

Komplette Liste: <http://www.mathematik.uni-stuttgart.de/preprints>

- 2010/012 *Effenberger, F.*: Stacked polytopes and tight triangulations of manifolds
- 2010/011 *Gyrfi, L.; Walk, H.*: Empirical portfolio selection strategies with proportional transaction costs
- 2010/010 *Kohler, M.; Krzyżak, A.; Walk, H.*: Estimation of the essential supremum of a regression function
- 2010/009 *Geisinger, L.; Laptev, A.; Weidl, T.*: Geometrical Versions of improved Berezin-Li-Yau Inequalities
- 2010/008 *Poppitz, S.; Stroppel, M.*: Polarities of Schellhammer Planes
- 2010/007 *Grundhfer, T.; Krinn, B.; Stroppel, M.*: Non-existence of isomorphisms between certain unitals
- 2010/006 *Hllig, K.; Hrner, J.; Hoffacker, A.*: Finite Element Analysis with B-Splines: Weighted and Isogeometric Methods
- 2010/005 *Kaltenbacher, B.; Walk, H.*: On convergence of local averaging regression function estimates for the regularization of inverse problems
- 2010/004 *Kühnel, W.; Solanes, G.*: Tight surfaces with boundary
- 2010/003 *Kohler, M.; Walk, H.*: On optimal exercising of American options in discrete time for stationary and ergodic data
- 2010/002 *Gulde, M.; Stroppel, M.*: Stabilizers of Subspaces under Similitudes of the Klein Quadric, and Automorphisms of Heisenberg Algebras
- 2010/001 *Leitner, F.*: Examples of almost Einstein structures on products and in cohomogeneity one
- 2009/008 *Griesemer, M.; Zenk, H.*: On the atomic photoeffect in non-relativistic QED
- 2009/007 *Griesemer, M.; Moeller, J.S.*: Bounds on the minimal energy of translation invariant n-polaron systems
- 2009/006 *Demirel, S.; Harrell II, E.M.*: On semiclassical and universal inequalities for eigenvalues of quantum graphs
- 2009/005 *Bächle, A.; Kimmerle, W.*: Torsion subgroups in integral group rings of finite groups
- 2009/004 *Geisinger, L.; Weidl, T.*: Universal bounds for traces of the Dirichlet Laplace operator
- 2009/003 *Walk, H.*: Strong laws of large numbers and nonparametric estimation
- 2009/002 *Leitner, F.*: The collapsing sphere product of Poincaré-Einstein spaces
- 2009/001 *Brehm, U.; Kühnel, W.*: Lattice triangulations of  $E^3$  and of the 3-torus
- 2008/006 *Kohler, M.; Krzyżak, A.; Walk, H.*: Upper bounds for Bermudan options on Markovian data using nonparametric regression and a reduced number of nested Monte Carlo steps
- 2008/005 *Kaltenbacher, B.; Schöpfer, F.; Schuster, T.*: Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems
- 2008/004 *Leitner, F.*: Conformally closed Poincaré-Einstein metrics with intersecting scale singularities
- 2008/003 *Effenberger, F.; Kühnel, W.*: Hamiltonian submanifolds of regular polytope
- 2008/002 *Hertweck, M.; Hofert, C.R.; Kimmerle, W.*: Finite groups of units and their composition factors in the integral group rings of the groups  $PSL(2, q)$
- 2008/001 *Kovarik, H.; Vugalter, S.; Weidl, T.*: Two dimensional Berezin-Li-Yau inequalities with a correction term
- 2007/006 *Weidl, T.*: Improved Berezin-Li-Yau inequalities with a remainder term
- 2007/005 *Frank, R.L.; Loss, M.; Weidl, T.*: Polya's conjecture in the presence of a constant magnetic field

- 2007/004 *Ekholm, T.; Frank, R.L.; Kovarik, H.:* Eigenvalue estimates for Schrödinger operators on metric trees
- 2007/003 *Lesky, P.H.; Racke, R.:* Elastic and electro-magnetic waves in infinite waveguides
- 2007/002 *Teufel, E.:* Spherical transforms and Radon transforms in Moebius geometry
- 2007/001 *Meister, A.:* Deconvolution from Fourier-oscillating error densities under decay and smoothness restrictions