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Abstract
In this paper we consider first an estimator of the residual variance treated by Evans (and Jones) (2005, 2008) and by Liitiäinen et al. (2008, 2010), based on first and second nearest neighbors given an independent and identically distributed sample. Its strong consistency and strong Cesàro consistency are shown under mere boundedness and square integrability, respectively, of the dependent variable Y. Moreover, in view of the local variance, a correspondingly modified estimator of local averaging (partitioning) type is proposed, and strong L_1-consistency (for bounded Y) and rate of convergence (for bounded X and Y under Lipschitz continuity of the regression and the local variance function) are established.

Key words: regression function, residual variance, local variance, partitioning estimation, nearest neighbors, strong consistency, rate of convergence.

AMS Subject classification: 62G05, 62G20.

1 Introduction

Let Y be a square integrable real valued random variable and let X be a d-dimensional random vector, taking values in the space $\mathbb{R}^d$. The task of regression analysis is to estimate Y given X, i.e., to find a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, such that $f(X)$ is a "good approximation" of Y, that is, $|f(X) - Y|$ has to be "small". The "closeness" of $f(X)$ to Y is typically measured by the so-called **mean squared error** of f,

$$\boldsymbol{E}\{(Y - f(X))^2\}.$$

It is well known that the regression function m minimizes this error (where $m := \boldsymbol{E}\{Y|X = x\}$),

$$V := \min_f \boldsymbol{E}\{(Y - f(X))^2\} = \boldsymbol{E}\{(Y - m(X))^2\}. \tag{1}$$

V, the so-called residual variance, is a measure of how close we can get to Y using any measurable function f. It indicates how difficult a regression problem is. Since the distribution of m, and therefore m, are unknown, one is interested in estimating V by use of data observations

$$D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}, \tag{2}$$

which are independent copies of (X, Y).
A related interesting problem is the estimation of the local variance (or conditional variance), defined as

$$\sigma^2(x) := \boldsymbol{E}\{(Y - m(X))^2|X = x\} = \boldsymbol{E}\{Y^2|X = x\} - m^2(x). \tag{3}$$

It holds

$$V = \boldsymbol{E}\{\sigma^2(X)\}. \tag{4}$$

Liitiäinen et al. [12], with generalization in [13], investigated an estimator of the residual variance V, introduced by Evans (and Jones) [5, 6], which is based on first and second nearest neighbors. They obtained mean square convergence under bounded conditional fourth moment of Y and convergence order $O(n^{-2/d})$ for $d \geq 2$ under finite suitable moments of X and under Lipschitz continuity of m. It simplifies an estimator given in Devroye et al. [3], based on first nearest neighbors. References for the estimation of the local variance function, incl. the case of fixed design, are Müller and Stadtmüller [15, 16], Stadtmüller and Tsybakov [23], Ruppert et al. [21], Härdle and Tsybakov [9], Spokoiny [22], Pan and Wang [20], Hall et al. [8], Müller et al. [17], Neumann [19], Munk et al. [18], Kohler [10], Brown and Levine [1], and Cai et al. [2].
In this paper, first we show strong consistency of the (global) residual variance estimation sequence of Evans (and Jones) [5, 6] and Liitiäinen et al. [12, 13], under boundedness of Y and show strong consistency of the sequence of arithmetic means in the general case $\boldsymbol{E}\{Y^2\} < \infty$ (Section 2).
In Section 3 for the estimation of the local variance function σ^2 on the basis of data (2), we propose an estimation sequence (σ_n^2) of local averaging, namely partitioning, type. It is a modification of the (global) residual vairance estimator and uses again first and second nearest neighbors. We show strong L_1-consistency, that is, $\int |\sigma_n^2(x) - \sigma^2|\mu(dx) \to 0$ $a.s.$, under mere boundedness of Y (μ denoting the distribution of X).
Finally, in Section 4 we establish its rate, imposing Lipschitz conditions on σ^2 and on m together with boundedness of X and Y.

2 Residual Variance Estimation

In the literature different paradigms how to construct nonparametric estimates are treated. Beside the least squares approach, local averaging paradigms are used, especially kernel estimates, partitioning estimates and k-th nearest neighbor estimates. A reference is Györfi et al. [7].
For given $i \in \{1, \ldots, n\}$, the first nearest neighbor of X_i among $X_1, \ldots, X_{i-1}$, $X_{i+1}, \ldots, X_n$ is defined as $X_{[N,1]}$ with

$$N[i, 1] := N_n[i, 1] := \underset{1 \leq j \leq n,\ j \neq i}{\arg\min}\ \rho(X_i, X_j), \tag{5}$$

here ρ is a metric (typically the Euclidean one) in $\mathbb{R}^d$. The k-th nearest neighbor of X_i among $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$ is defined as $X_{N[i,k]}$ via generalization of definition (5):

$$N[i,k] := N_n[i,k] := \underset{1 \le j \le n,\ j \ne i,\ j \notin \{N[i,1], \dots, N[i,k-1]\}}{\arg\min} \rho(X_i, X_j), \tag{6}$$

by removing the preceeding neighbors. If ties occur, a possibility to break them is given by taking the minimal index or by adding independent components Z_i, uniformly distributed on $[0,1]$, to the observation vectors X_i (see [7], pp. 86, 87). The latter possibility to break ties allow us to assume throughout the paper that ties occur with probability zero.

Hence, we get a reorder of the data according to increasing values of the distance of the variable X_j ($j \in \{1, \dots, n\} \setminus \{i\}$) from the variable X_i ($i = 1, \dots, n$). Correspondingly to that, we get also a new order for the variables Y_j :

$$\left(X_{N[i,1]},\ Y_{N[i,1]}\right),\ \dots, \left(X_{N[i,k]},\ Y_{N[i,k]}\right),\ \dots, \left(X_{N[i,n-1]},\ Y_{N[i,n-1]}\right).$$

In the following $N[i,1]$ and $N[i,2]$ will be used.

For the residual variance V, Evans (and Jones) [5, 6] introduced and Liitiäinen et al. [12, 13] analyzed (and generalized) the estimator

$$V_n = \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - Y_{N[i,1]}\right)\left(Y_i - Y_{N[i,2]}\right), \tag{7}$$

in view of square mean consistency and rate of convergence.

We shall establish strong consistency.

Theorem 2.1 *If $|Y| \le L$ for some $L \in R_+$, then*

$$V_n \to V \quad a.s. \quad (n \to \infty).$$

The proof is based on the McDiarmid inequality (see, e.g., [7], Theorem A.2) and properties of nearest neighbors (see [7], Lemma 6.1 and Corollary 6.1 together with Lemma 6.3). To make the paper more self-contained, we state them in the following lemmas.

Lemma 2.2 *(McDiarmid inequality) Let $Z_1, \dots, Z_n$ be independent random variables taking values in a set A and assume that $f : A^n \to \mathbb{R}$ satisfies*

$$\sup_{z_1, \dots, z_n,\ z_i' \in A} |f(z_1, \dots, z_n) - f(z_1, \dots, z_{i-1}, z_i', z_{i+1}, \dots, z_n)| \le c_i, \quad 1 \le i \le n.$$

Then, for all $\epsilon > 0$,

$$\boldsymbol{P}\{f(Z_1, \dots, Z_n) - \boldsymbol{E} f(Z_1, \dots, Z_n) \ge \epsilon\} \le e^{-\frac{2\epsilon^2}{\sum_{i=1}^n c_i^2}},$$

and

$$\boldsymbol{P}\{\boldsymbol{E} f(Z_1, \dots, Z_n) - f(Z_1, \dots, Z_n) \ge \epsilon\} \le e^{-\frac{2\epsilon^2}{\sum_{i=1}^n c_i^2}}.$$

Lemma 2.3 *If $k_n/n \to 0$, then*

$$\|X_{N[1,k_n]} - X_1\| \to 0 \quad a.s.$$

Lemma 2.4 *Under the assumption that ties occur with probability zero,*

a)

$$\sum_{i=1}^{n} 1_{\{X \text{ is among the } k \text{ nearest neighbors of } X_i \text{ in } \{X_1, \dots, X_{i-1}, X, X_{i+1}, \dots, X_n\}\}}$$
$$\le k\gamma_d \quad a.s. \quad (k \le n),$$

b) for any integrable function f and any $k \leq n-1$,

$$\sum_{j=1}^{k} \boldsymbol{E}\{|f(X_{N[1,j]})|\} \leq k\gamma_d \boldsymbol{E}\{|f(X_1)|\},$$

Here $\gamma_d < \infty$ depends only on d.

Proof of Theorem 2.1 In the first step we show

$$\boldsymbol{E}V_n \to V \tag{8}$$

(asymptotic unbiasedness), using only square integrability of Y, compare [13], proof of Theorem 2.2.

With the notations

$$\begin{aligned} b_{i,j} &= m(X_i) - m(X_j) \\ r_i &= Y_i - m(X_i), \end{aligned} \tag{9}$$

we can write, according to [12] and [13]:

$$\begin{aligned} &\boldsymbol{E}\left\{Y_i - Y_{N[i,1]}\right)\left(Y_i - Y_{N[i,2]}\right\} = \\ &\boldsymbol{E}\left\{b_{i,N[i,1]}\left(r_i - r_{N[i,2]}\right)\right\} + \boldsymbol{E}\left\{b_{i,N[i,2]}\left(r_i - r_{N[i,1]}\right)\right\} \\ &+\boldsymbol{E}\left\{\left(r_i - r_{N[i,1]}\right)\left(r_i - r_{N[i,2]}\right)\right\} + \boldsymbol{E}\left\{b_{i,N[i,1]}b_{i,N[i,2]}\right\}. \end{aligned}$$

As shown in [12] and [13] via conditioning with respect to $X_1, \ldots, X_n$,

$$\boldsymbol{E}\left\{b_{i,N[i,1]}\left(r_i - r_{N[i,2]}\right)\right\} = \boldsymbol{E}\left\{b_{i,N[i,2]}\left(r_i - r_{N[i,1]}\right)\right\} = 0,$$

and

$$\boldsymbol{E}\left\{\left(r_i - r_{N[i,1]}\right)\left(r_i - r_{N[i,2]}\right)\right\} = \boldsymbol{E}\{r_i^2\} = \boldsymbol{E}\{(Y_i - m(X_i))^2\} = V.$$

Further

$$\left|\boldsymbol{E}\left\{b_{i,N[i,1]}b_{i,N[i,2]}\right\}\right| \leq \boldsymbol{E}\left\{|(m(X_i) - m(X_{N[i,1]})||m(X_i) - m(X_{N[i,2]})|\right\}.$$

Thus, because the X_i's are identically distributed,

$$\begin{aligned} &|\boldsymbol{E}V_n - V| \leq \boldsymbol{E}\left\{|(m(X_1) - m(X_{N[1,1]})||m(X_1) - m(X_{N[1,2]})|\right\} \\ \leq\ & \frac{1}{2}\boldsymbol{E}\left\{|m(X_1) - m(X_{N[1,1]})|^2\right\} + \frac{1}{2}\boldsymbol{E}\left\{|m(X_1) - m(X_{N[1,2]})|^2\right\}. \end{aligned}$$

Because the set of continuous functions on $\mathbb{R}^d$ with compact support is dense in $L_2(\mu)$ (see, e.g., [4], Chapter 4, Section 8.19, or [7], Theorem A.1), for an arbitrary $\epsilon > 0$ one can choose a continuous function $\widetilde{m}$ with compact support such that $\boldsymbol{E}\{|m(X_1) - \widetilde{m}(X_1)|^2\} \leq \epsilon$. Then

$$\begin{aligned} & \boldsymbol{E}\{|m(X_1) - m(X_{N[1,1]})|\} \\ \leq\ & 3\boldsymbol{E}\{|(m - \widetilde{m})(X_1)|^2\} + 3\boldsymbol{E}\{|(m - \widetilde{m})(X_{N[1,1]})|^2\} \\ & +3\boldsymbol{E}\{|(\widetilde{m}(X_1) - \widetilde{m}(X_{N[1,1]})|^2\}. \end{aligned}$$

By Lemma 2.3 (with $k_n = 1$) and continuity of $\widetilde{m}$, one has

$$\widetilde{m}(X_{N[1,1]}) \to \widetilde{m}(X_1) \quad a.s.,$$

thus, by boundedness of $\widetilde{m}$,

$$\boldsymbol{E}\{|\widetilde{m}(X_1) - \widetilde{m}(X_{N[1,1]})|^2\} \to 0.$$

Further, by Lemma 2.4b,

$$\begin{aligned} & \boldsymbol{E}\{|(m - \widetilde{m})(X_{N[1,1]})|^2\} \\ \leq\ & \gamma_d \boldsymbol{E}\{|(m - \widetilde{m})(X_1)|\} \leq \gamma_d \epsilon. \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \boldsymbol{E}\{|m(X_1) - m(X_{N[1,1]})|^2\} \leq 3(1 + \gamma_d)\epsilon,$$

thus

$$\boldsymbol{E}\{|m(X_1) - m(X_{N[1,1]})|^2\} \to 0.$$

Analogously one obtains $\boldsymbol{E}\{|m(X_1) - m(X_{N[1,2]})|^2\} \to 0$. Thus

$$\boldsymbol{E}\{|m(X_1) - m(X_{N[1,1]})||m(X_1) - m(X_{N[1,2]})|\} \to 0, \tag{10}$$

and (8) is obtained.
In the second step we show

$$V_n - \boldsymbol{E}V_n \to 0 \quad a.s. \tag{11}$$

Set

$$T_n := \sum_{i=1}^{n} (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}).$$

Now in view of an application of Lemma 2.2, let $(X_1, Y_1), \ldots, (X_n, Y_n)$, $(X_1', Y_1'), \ldots, (X_n', Y_n')$ be independent and identically distributed $(d+1)$-dimensional random vectors. For fixed $j \in \{1, \ldots, n\}$ replace (X_j, Y_j) by (X_j', Y_j'), which leads to $T_{n,j}$. Noticing $|Y_i| \leq L$, we have

$$|T_n - T_{n,j}| \leq 8L^2 + 8L^2 \cdot 2 \cdot 2\gamma_d = 8(1 + 4\gamma_d)L^2, \tag{12}$$

where the first term of the right-hand side results from summand $i = j$ and the second term results from summands $i \in \{1, \ldots, n\} \setminus \{j\}$, because replacement of X_j by X_j' has an influence on the first and second nearest neighbors of some, but at most $2\gamma_d$ (by Lemma 2.4 a), of the random vectors $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n$. By Lemma 2.2, for each $\epsilon > 0$ we obtain

$$\begin{aligned} & \boldsymbol{P}\{|V_n - \boldsymbol{E}V_n| \geq \epsilon\} \\ = \; & \boldsymbol{P}\{|T_n - \boldsymbol{E}T_n| \geq \epsilon n\} \\ \leq \; & 2e^{-2\epsilon^2 n^2 / n(8(1+4\gamma_d)L^2)^2}, \end{aligned}$$

thus (11) by the Borel-Cantelli lemma.
(8) and (11) yield the assertion.

■

The following theorem states that the boundedness assumption in Theorem 2.1 on Y can be omitted if for estimation of V the sequence $((V_1 + \ldots, V_n)/n)$ of arithmetic means insted of (V_n) is used.

Theorem 2.5 *In the general case* $\boldsymbol{E}\{Y^2\} < \infty$,

$$\frac{V_1 + \ldots, V_n}{n} \to V \quad a.s.$$

(strong Cesàro consistency of (V_n)).

It remains an open problem whether $V_n \to V$ *a.s.* if $\boldsymbol{E}\{Y^2\} < \infty$.
For the proof of Theorem 2.5 we shall use an Efron-Stein inequality (Lemma 2.6, compare [7], Theorem A.3).

Lemma 2.6 *Let $Z_1, \ldots, Z_n, \widetilde{Z}_1, \ldots, \widetilde{Z}_n$ be independent m-dimensional random vectors where the two random vectors Z_k and $\widetilde{Z}_k$ have the same distribution ($k = 1, \ldots, n$). For measurable $f : \mathbb{R}^{m \cdot n} \to \mathbb{R}$ assume that $f(Z_1, \ldots, Z_n)$ is square integrable. Then*

$$\begin{aligned} & \boldsymbol{Var}\{f(Z_1, \ldots, Z_n)\} \\ \leq \; & \frac{1}{2} \sum_{k=1}^{n} \boldsymbol{E}\left\{|f(Z_1, \ldots, Z_k, \ldots, Z_n) - f(Z_1, \ldots, \widetilde{Z}_k, \ldots, Z_n)|^2\right\}. \end{aligned}$$

Proof of Theorem 2.5 For a real random variable U we set

$$U^{[c]} := U1_{\{|U|\le c\}} + c1_{\{U>c\}} - c1_{\{U<-c\}}, \quad c>0.$$

First we show

$$\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - Y_{N[i,1]}\right)\left(Y_i - Y_{N[i,2]}\right) - \frac{1}{n}\sum_{i=1}^{n} V_{n,i} \to 0 \quad a.s.,$$

where

$$V_{n,i} := \left(Y_i^{[\sqrt{n}]} - Y_{N[i,1]}^{[\sqrt{n}]}\right)\left(Y_i^{[\sqrt{n}]} - Y_{N[i,2]}^{[\sqrt{n}]}\right).$$

Because $\boldsymbol{E}\{Y\}^2 < \infty$, *a.s.* $Y_i = Y_i^{[\sqrt{i}]}$ for i sufficiently large, say, $i \ge M$ (random). For $i \in \{M, M+1, \ldots, n\}$, *a.s.* $Y_i = Y_i^{[\sqrt{n}]}$. By Lemma 2.4a, for $p \in \{1, \ldots, M\}$ one has $N[i,1] = p$ for at most γ_d indices $\in \{1, \ldots, n\}$ and $N[i,2] = p$ for at most $2\gamma_d$ indices $i \in \{1, \ldots, n\}$. Thus *a.s.*

$$\left(Y_i - Y_{N[i,1]}\right)\left(Y_i - Y_{N[i,2]}\right) \ne \left(Y_i^{[\sqrt{n}]} - Y_{N[i,1]}^{[\sqrt{n}]}\right)\left(Y_i^{[\sqrt{n}]} - Y_{N[i,2]}^{[\sqrt{n}]}\right)$$

for at most $(1+3\gamma_d)M$ indices $\in \{1, \ldots, n\}$, which yields the assertion.
Therefore it suffices to show

$$\frac{1}{n}\sum_{l=1}^{n}\left(\frac{1}{l}\sum_{i=1}^{l} V_{l,i}\right) \to V \quad a.s. \tag{13}$$

In the second step we show

$$\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{E} V_{n,i} \to V. \tag{14}$$

With $m^{(n)}(x) := \boldsymbol{E}\{Y^{[\sqrt{n}]} | X = x\}$ we have

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{E} V_{n,i} = \boldsymbol{E} V_{n,1} \\ = \; & \boldsymbol{E}\left\{(Y^{[\sqrt{n}]} - m^{(n)}(X))^2\right\} \\ + \; & \boldsymbol{E}\left\{\left(m^{(n)}(X_1) - m^{(n)}(X_{N[1,1]})\right)\left(m^{(n)}(X_1) - m^{(n)}(X_{N[1,2]})\right)\right\}, \end{aligned}$$

the latter according to Liitiäinen et al. [12, 13]. By $\boldsymbol{E}\{Y^2\} < \infty$ and the dominated convergence theorem, $\int |m^{(n)}(x) - m(x)|^2 \mu(dx) \to 0$ and thus $\boldsymbol{E}\{(Y^{[\sqrt{n}]} - m^{(n)}(X))^2\} \to V$. Further m and also $m^{(n)}$ can be approximated by a continuous function $\widetilde{m}$ with compact support such that for each $\epsilon > 0$ an index $n_0(\epsilon)$ exists with $\boldsymbol{E}\{|m(X) - \widetilde{m}(X)|^2\} \le \epsilon$ and also

$$\boldsymbol{E}\{|m^{(n)}(X) - \widetilde{m}(X)|^2\} \le \epsilon \text{ for } n \ge n_0(\epsilon).$$

Then we obtain

$$\begin{aligned} & \boldsymbol{E}\left\{\left|m^{(n)}(X_1) - m^{(n)}(X_{N[1,1]})\right|^2\right\} \\ \le \; & 3\boldsymbol{E}\{|(m^{(n)} - \widetilde{m})(X_1)|^2\} + \\ & 3\boldsymbol{E}\{|(m^{(n)} - \widetilde{m})(X_{N[1,1]})|^2\} + 3\boldsymbol{E}\{|\widetilde{m}(X_1) - \widetilde{m}(X_{N[1,1]})|^2\} \\ \le \; & 3\epsilon + 3\gamma_d\epsilon + o(1), \end{aligned}$$

the latter as in the proof of Theorem 2.1. Therefore

$$\boldsymbol{E}\left\{\left|m^{(n)}(X_1) - m^{(n)}(X_{N[1,1]})\right|^2\right\} \to 0$$

and correspondingly

$$\boldsymbol{E}\left\{\left|m^{(n)}(X_1) - m^{(n)}(X_{N[1,2]})\right|^2\right\} \to 0,$$

thus

$$\boldsymbol{E}\left\{\left(m^{(n)}(X_1) - m^{(n)}(X_{N[1,1]})\right)\left(m^{(n)}(X_1) - m^{(n)}(X_{N[1,2]})\right)\right\} \to 0,$$

and (14) is obtained as well as

$$\frac{1}{n}\sum_{l=1}^{n}\left(\frac{1}{l}\sum_{i=1}^{l}\boldsymbol{E}V_{l,i}\right)\to V. \tag{15}$$

In the second step we show

$$\frac{1}{n}\sum_{l=1}^{n}\left(\frac{1}{l}\sum_{i=1}^{l}(V_{l,i}-\boldsymbol{E}V_{l,i})\right)\to 0 \quad a.s. \tag{16}$$

It suffices to show

$$\sum\frac{\boldsymbol{Var}\left\{\sum_{i=1}^{n}V_{n,i}\right\}}{n^3}<\infty, \tag{17}$$

for this implies

$$\sum\frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}(V_{n,i}-\boldsymbol{E}V_{n,i})\right)^2<\infty \quad a.s.$$

and, by the Cauchy-Schwarz inequality and the Kronecker lemma,

$$\begin{aligned}&\left|\frac{1}{n}\sum_{l=1}^{n}\left(\frac{1}{l}\sum_{i=1}^{l}\left(V_{l,i}-\boldsymbol{E}V_{l,i}\right)\right)\right|^2\\ \leq\ &\frac{1}{n}\sum_{l=1}^{n}\left|\frac{1}{l}\sum_{i=1}^{l}\left(V_{l,i}-\boldsymbol{E}V_{l,i}\right)\right|^2\to 0 \quad a.s.\end{aligned}$$

We shall show

$$\boldsymbol{Var}\left\{\sum_{i=1}^{n}V_{n,i}\right\}\leq cn\boldsymbol{E}\left\{\left(Y^{[\sqrt{n}]}\right)^4\right\},\ n\in\mathbb{N} \tag{18}$$

for a suitable finite constant c. This, together with $\boldsymbol{E}\{Y^2\}<\infty$, implies (17), because, as is well known (see, e.g., [14], Section 17.3), $\boldsymbol{E}|U|<\infty$ for a real variable U implies $\sum\boldsymbol{E}\left\{\left(U^{[n]}\right)^2\right\}/n^2<\infty$.
We prove (18) by using the Efron-Stein inequality (Lemma 2.6).
Let $n\geq 2$ be fixed. Replacement of (X_j,Y_j) by (X'_j,Y'_j) for fixed $j\in\{1,\dots,n\}$ (where $(X_1,Y_1),\dots(X_n,Y_n),(X'_1,Y'_1),\dots,(X'_n,Y'_n)$ are independent and identically distributed) leads from $T_n:=\sum_{i=1}^{n}V_{n,i}$, $N[j,1]$ and $N[j,2]$ to $T_{n,j}$, $N'[j,1]$ and $N'[j,2]$, respectively.
We obtain

$$|T_n-T_{n,j}|\leq A_{n,j}+B_{n,j}+C_{n,j}+D_{n,j}+E_{n,j}+F_{n,j}$$

where with $Z_i=Y_i^{[\sqrt{n}]}$, $Z'_j=Y_j'^{[\sqrt{n}]}$, $Z=Y^{[\sqrt{n}]}$

$$A_{n,j}=\sum_{\substack{l,\ q\in\{1,\dots,n\}\setminus\{j\}\\ l\neq q}}|Z_j-Z_l||Z_j-Z_q|1_{\{N[j,1]=l\}}1_{\{N[j,2]=q\}},$$

$$B_{n,j}=\sum_{\substack{l,\ q\in\{1,\dots,n\}\setminus\{j\}\\ l\neq q}}|Z'_j-Z_l||Z'_j-Z_q|1_{\{N'[j,1]=l\}}1_{\{N'[j,2]=q\}},$$

$$C_{n,j}=\sum_{\substack{i,\ q\in\{1,\dots,n\}\setminus\{j\}\\ i\neq q}}|Z_i-Z_j||Z_i-Z_q|1_{\{N[i,1]=j\}}1_{\{N[i,2]=q\}},$$

$$D_{n,j}=\sum_{\substack{i,\ q\in\{1,\dots,n\}\setminus\{j\}\\ i\neq q}}|Z_i-Z'_j||Z_i-Z_q|1_{\{N'[i,1]=j\}}1_{\{N'[i,2]=q\}},$$

$$E_{n,j}=\sum_{\substack{i,\ l\in\{1,\dots,n\}\setminus\{j\}\\ i\neq l}}|Z_i-Z_l||Z_i-Z_j|1_{\{N[i,1]=l\}}1_{\{N[i,2]=j\}},$$

$$F_{n,j} = \sum_{\substack{i,\ l \in \{1,\dots,n\}\setminus\{j\} \\ i \neq l}} |Z_i - Z_l||Z_i - Z'_j| 1_{\{N'[i,1]=l\}} 1_{\{N'[i,2]=j\}}.$$

Thus

$$|T_n - T_{n,j}|^2 \leq 6(A_{n,j}^2 + B_{n,j}^2 + C_{n,j}^2 + D_{n,j}^2 + E_{n,j}^2 + F_{n,j}^2).$$

By the Cauchy-Schwarz inequality applied to the sums defining $A_{n,j}, \dots,$ $F_{n,j}$ and by the inequality $|a-b|^2|a-c|^2 \leq 8(a^4 + b^4 + c^4)$ we obtain

$$\boldsymbol{E} \sum_{j=1}^{n} |T_n - T_{n,j}|^2 \leq 6 \cdot 6 \cdot 8 \boldsymbol{E} \sum_{\substack{j,\ l,\ q \in \{1,\dots,n\} \\ j \neq l,\ l \neq q,\ q \neq j}} (Z_j^4 + Z_l^4 + Z_q^4) 1_{\{N[j,1]=l\}} 1_{\{N[j,2]=q\}}.$$

As to the term concerning Z_j^4 we sum with respect to l and q and for the corresponding expected final sum we obtain the bound $n\boldsymbol{E}\left\{Z^4\right\}$. As to the term Z_l^4 we sum with respect to q, then with respect to j using Lemma 2.4a, and for the corresponding expected final sum we obtain the bound $\gamma_d n \boldsymbol{E}\left\{Z^4\right\}$. As to the term Z_q^4 we sum with respect l, then with respect to j using Lemma 2.4b and for the corresponding expected final sum we obtain the bound $2\gamma_d n\boldsymbol{E}\left\{Z^4\right\}$.
Therefore, by Lemma 2.6,

$$\boldsymbol{Var}(T_n) \leq \frac{1}{2} \cdot 6 \cdot 6 \cdot 8 \cdot (1 + 3\gamma_d) n \boldsymbol{E}\left\{\left(Y^{[\sqrt{n}]}\right)^4\right\},$$

i.e., (18). Thus (16) is obtained, which together with (15) implies (13). ∎

3 Local Variance Estimation: Strong Consistency

V_n in (7) as an estimator of $V = \boldsymbol{E}\{(Y - m(X))^2\}$ was treated in Section 2. In this section our aim is to give an estimator of the local variance function σ^2 in (3). Recall the relation between the residual and the local variance function in (4).
Our proposal for an appropriate estimator of σ^2 is

$$\sigma_n^2(x) := \frac{\sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}) 1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)},\ x \in \mathbb{R}^d \tag{19}$$

where $\mathcal{P}_n = \{A_{n,1}, A_{n,2}, \dots\}$ is a partition of $\mathbb{R}^d$ consisting of Borel sets $A_{n,j} \subset \mathbb{R}^d$, and where the notation $A_n(x)$ is used for the $A_{n,j}$ containing x. In this sense we localize the global expression in V_n by local averaging, in particular by partitioning. Analogously a kernel type estimator could be treated. The next theorem deals with strong consistency of the local variance estimator.

Theorem 3.1 *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ with $\mathcal{P}_n = \{A_{n,1}, A_{n,2}, \dots\}$ be a sequence of partitions of $\mathbb{R}^d$ such that for each sphere S centered at the origin*

$$\lim_{n\to\infty} \max_{j: A_{n,j} \cap S \neq \emptyset} \mathit{diam}\ A_{n,j} \to 0 \tag{20}$$

and, for some $\rho = \rho(S) \in (0, \frac{1}{2})$

$$\#\{j : A_{n,j} \cap S \neq \emptyset\} \sim n^\rho. \tag{21}$$

Finally, let $|Y| \leq L$ for some $L \in \mathbb{R}_+$. Then

$$\int |\sigma_n^2(x) - \sigma^2(x)| \mu(dx) \to 0 \quad a.s.$$

Set now

$$\sigma_n^{2*}(x) := \frac{\sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}) 1_{A_n(x)}(X_i)}{n\mu(A_n(x))}. \tag{22}$$

For the proof of Theorem 3.1 we need Lemma 3.3, which is based on Lemma 3.2 and the McDiarmid inequality (Lemma 2.2). Lemma 3.2 itself is based on the Efron-Stein inequality in Lemma 2.6.

Lemma 3.2 *Under (20) and (21), for each sphere S centered at* 0

$$\boldsymbol{E}\left\{\int_S |\sigma^2(x) - \sigma_n^{2*}(x)|\mu(dx)\right\} \to 0.$$

Proof One has

$$\begin{aligned}
&\boldsymbol{E}\left\{\int_S |\sigma^2(x) - \sigma_n^{2*}(x)|\mu(dx)\right\} \\
\leq\ & \int_S |\sigma^2(x) - \boldsymbol{E}\sigma_n^{2*}(x)|\mu(dx) + \boldsymbol{E}\left\{\int_S |\boldsymbol{E}\sigma_n^{2*}(x) - \sigma_n^{2*}(x)|\mu(dx)\right\} \\
\leq\ & K_n + M_n
\end{aligned}$$

First $K_n \to 0$ will be shown. According to Liitiäinen et al. [12, 13] one has

$$\begin{aligned}
&\boldsymbol{E}\{(Y_1 - Y_{N[1,1]})(Y_1 - Y_{N[1,2]})|X_1 = z\} \\
=\ & \sigma^2(z) + \boldsymbol{E}\{(m(X_1) - m(X_{N[1,1]}))(m(X_1) - m(X_{N[1,2]}))|X_1 = z\},
\end{aligned}$$

thus

$$\begin{aligned}
&\boldsymbol{E}\sigma_n^{2*}(x) \\
=\ & \int \frac{\sigma^2(z) 1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz) \\
&+ \int \frac{\boldsymbol{E}\{(m(X_1) - m(X_{N[1,1]}))(m(X_1) - m(X_{N[1,2]}))|X_1 = z\} 1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz).
\end{aligned}$$

Notice

$$\begin{aligned}
&\int \left[\int \frac{\boldsymbol{E}\{|m(X_1) - m(X_{N[1,1]})||m(X_1) - m(X_{N[1,2]})||X_1 = z\} 1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz)\right]\mu(dx) \\
=\ & \int \left[\int \frac{\boldsymbol{E}\{|m(X_1) - m(X_{N[1,1]})||m(X_1) - m(X_{N[1,2]})||X_1 = z\} 1_{A_n(z)}(x)}{\mu(A_n(x))}\mu(dx)\right]\mu(dz) \\
\leq\ & \boldsymbol{E}\{|m(X_1) - m(X_{N[1,1]}||m(X_1) - m(X_{N[1,2]})|\} \\
&\to 0
\end{aligned}$$

by (10). Moreover,

$$\int \left|\sigma^2(x) - \int \frac{\sigma^2(z) 1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz)\right|\mu(dx) \to 0.$$

For, because of $\int \sigma^2(x)\mu(dx) < \infty$, as in the proof of Theorem 2.1 for each $\epsilon > 0$ one can choose a continuous function $\widetilde{\sigma}^2$ with compact support such that

$$\int |\sigma^2(x) - \widetilde{\sigma}^2(x)|\mu(dx) < \epsilon,$$

further

$$\begin{aligned}
&\int \left|\int \frac{\sigma^2(z) 1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz) - \int \frac{\widetilde{\sigma}^2(z) 1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz)\right|\mu(dx) \\
\leq\ & \int \left|\sigma^2(z) - \widetilde{\sigma}^2(z)\right|\mu(dz) < \epsilon,
\end{aligned}$$

and one then notices

$$\int_S \left| \widetilde{\sigma}^2(x) - \int \frac{\widetilde{\sigma}^2(z) 1_{A_n(x)}(z)}{\mu(A_n(x))} \mu(dz) \right| \mu(dx) \to 0$$

because of uniform continuity of $\widetilde{\sigma}$ and (20). Therefore $K_n \to 0$.
Now M_n will be treated. Set $J_n := \{j : \; A_{n,j} \cap S \neq \emptyset\}$ and $l_n := \# J_n$.

$$\begin{aligned}
M_n &= \sum_{j \in J_n} \boldsymbol{E} \left\{ \int_{A_{n,j}} \left| \frac{\sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}) 1_{A_{n,j}}(X_i)}{n\mu(A_{n,j})} \right. \right. \\
&\quad \left. \left. - \boldsymbol{E} \frac{\sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}) 1_{A_{n,j}}(X_i)}{n\mu(A_{n,j})} \right| \mu(dx) \right\} \\
&\leq \frac{1}{n} \sum_{j \in J_n} \boldsymbol{E} \left\{ \left| \sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}) 1_{A_{n,j}}(X_i) \right. \right. \\
&\quad \left. \left. - \boldsymbol{E} \sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}) 1_{A_{n,j}}(X_i) \right| \right\} \\
&\leq \frac{1}{n} \sum_{j \in J_n} \sqrt{ \boldsymbol{Var} \left\{ \sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}) 1_{A_{n,j}}(X_i) \right\} } \\
&\leq \frac{l_n}{n} \sqrt{ \frac{n}{2} (8L^2 + 8L^2 \cdot 2 \cdot 2\gamma_d)^2 } \\
&\quad \text{(by Lemma 2.6 and the derivation of (12))} \\
&\leq 4\sqrt{2}(1 + 4\gamma_d) L^2 \frac{l_n}{\sqrt{n}} \to 0 \quad \text{(by (11))}.
\end{aligned}$$

Thus the assertion is obtained. ∎

Lemma 3.3 *Assume (20) and (21). Let S be an arbitrary sphere centered at 0. Then a constant $c > 0$ exists such that for each $\epsilon > 0$*

$$\boldsymbol{P} \left\{ \int_S |\sigma^2(x) - \sigma_n^{2*}(x)| \mu(dx) > 2\epsilon \right\} \leq e^{-\epsilon^2 c n^{1-2\rho}}$$

for n sufficiently large.

Proof We follow the argument in the proof of Lemma 23.2 in [7]. One has

$$\begin{aligned}
& |\sigma^2(x) - \sigma_n^{2*}(x)| \\
= \; & \boldsymbol{E} |\sigma^2(x) - \sigma_n^{2*}(x)| + (|\sigma^2(x) - \sigma_n^{2*}(x)| - \boldsymbol{E} |\sigma^2(x) - \sigma_n^{2*}(x)|).
\end{aligned}$$

But $\int_S \boldsymbol{E} |\sigma^2(x) - \sigma_n^{2*}(x)| \mu(dx) \to 0$ due to Lemma 3.2.
Now, in view of an application of McDiarmid's inequality (Lemma 2.2) replacing (X_i, Y_i) by (X_i', Y_i') as in the proof of Theorem 2.1 leads from $\sigma_n^{2*}(x)$ to $\sigma_{n,j}^{2*}(x)$, $(j \in \{1, \ldots, n\})$, where, correspondingly to (12),

$$|\sigma_n^{2*}(x) - \sigma_{n,j}^{2*}(x)| \leq \frac{8(1 + 4\gamma_d) L^2}{n\mu(A_n(x))}.$$

Thus

$$\begin{aligned}
& \left| \int_S |\sigma^2(x) - \sigma_n^{2*}(x)| \mu(dx) - \int |\sigma^2(x) - \sigma_{n,j}^{2*}(x)| \mu(dx) \right| \\
= \; & \left| \int_S \left(|\sigma^2(x) - \sigma_n^{2*}(x)| - |\sigma^2(x) - \sigma_{n,j}^{2*}(x)| \right) \mu(dx) \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_S |\sigma_n^2(x) - \sigma_{n,j}^{2*}(x)|\mu(dx) \quad (j=1,\dots,n)\\
&\quad \text{(due to the triangle inequality } |a-b| \ge ||a|-|b||)\\
&\le \frac{8(1+4\gamma_d)L^2}{n}\int_S \frac{1}{\mu(A_n(x))}\mu(dx)\\
&\le \frac{8(1+4\gamma_d)L^2}{n} l_n,
\end{aligned}
$$

where $l_n := \#\{j : A_{n,j} \cap S \neq \emptyset\}$.
Now, using Lemma 2.2, for arbitrary $\epsilon > 0$

$$
\begin{aligned}
&\boldsymbol{P}\left\{\int_S \left(\int |\sigma^2(x) - \sigma_n^{2*}(x)|\mu(dx) - \boldsymbol{E}\int_S |\sigma^2(x) - \sigma_n^{2*}(x)|\mu(dx)\right) > \epsilon\right\}\\
&\le e^{-2\epsilon^2/n\frac{[8(1+4\gamma_d)L^2]^2}{n^2}l_n^2}\\
&\le e^{-\epsilon^2 c n^{1-2\rho}}.
\end{aligned}
$$

with some $c > 0$. Therefore, because of $\int_S \boldsymbol{E}|\sigma^2(x) - \sigma_n^{2*}(x)|\mu(dx) < \epsilon$ for n large enough,

$$
\boldsymbol{P}\left\{\int_S |\sigma^2(x) - \sigma_n^{2*}(x)|\mu(dx) > 2\epsilon\right\} \le e^{-\epsilon^2 c n^{1-2\rho}}
$$

for n sufficiently large. ■

Proof of Theorem 3.1 Because Y is bounded, for an arbitrary $\epsilon > 0$ one can choose a sphere S centered at 0, such that

$$
\int_{S^c} |\sigma_n^2(x) - \sigma^2(x)|\mu(dx) \le \epsilon.
$$

Therefore it sufficies to show $\int_S |\sigma_n^{2*}(x) - \sigma^2(x)|\mu(dx) \to 0$ *a.s.* for each sphere S centered at 0. One obtains

$$
\begin{aligned}
&\int_S |\sigma_n^2(x) - \sigma^2(x)|\mu(dx)\\
&\le \int_S |\sigma_n^2(x) - \sigma_n^{2*}(x)|\mu(dx) + \int_S |\sigma_n^{2*}(x) - \sigma^2(x)|\mu(dx)\\
&\le G_n + D_n.
\end{aligned}
$$

But $D_n \to 0$ due to Lemma 3.3 and the Borel-Cantelli lemma.
Now, concerning G_n, similarly to the argument in [7], p. 465,

$$
\begin{aligned}
&\int |\sigma_n^{2*}(x) - \sigma_n^2(x)|\mu(dx)\\
&\le \int \left| \frac{\sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]})1_{A_n(x)}(X_i)}{n\mu(A_n(x))}\right.\\
&\quad \left. - \frac{\sum_{i=1}^n (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]})1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)}\right| \mu(dx)\\
&\le 4L^2 \int \sum_{i=1}^n 1_{A_n(x)}(X_i) \left| \frac{1}{n\mu(A_n(x))} - \frac{1}{\sum_{i=1}^n 1_{A_n(x)}(X_i)}\right| \mu(dx)\\
&\le 4L^2 \left| \sum_{i=1}^n \frac{1_{A_n(x)}(X_i)}{n\mu(A_n(x))} - 1\right| \mu(dx) \to 0 \quad a.s.\\
&\quad \text{(due to (20) and (21)).}
\end{aligned}
$$

■

4 Rate of Convergence

In this section we establish a rate of convergence for the estimate of the local variance defined in (19). The rate corresponds to the rate obtained in classical regression estimation ([7], Theorems 4.3 and 3.2).

Theorem 4.1 *Let $\mathcal{P}_n$ be a cubic partition of $\mathbb{R}^d$ with side length h_n of the cubes ($n \in \mathbb{N}$). Assume that X and Y are bounded. Moreover, assume the Lipschitz conditions*

$$|\sigma^2(x) - \sigma^2(t)| \leq C\|x - t\|, \quad x, t \in \mathbb{R}^d, \tag{23}$$

and

$$|m(x) - m(t)| \leq D\|x - t\|, \quad x, t \in \mathbb{R}^d \tag{24}$$

($C,\ D \in \mathbb{R}_+$, $\|\ \|$ denoting the Euclidean norm).
Then, with

$$h_n \sim n^{-\frac{1}{d+2}}$$

for the estimate (19) one gets

$$\boldsymbol{E} \int |\sigma_n^2(x) - \sigma(x)|\mu(dx) = O\left(n^{-\frac{1}{d+2}}\right).$$

For the proof of Theorem 4.1 the following lemma will be used.

Lemma 4.2 *Assume that X is bounded. Then for some finite constant c,*

$$\boldsymbol{E}\{\|X_{N[1,1]} - X_1\|^2\} \leq cn^{-2/\max\{d,2\}},$$

$$\boldsymbol{E}\{\|X_{N[1,2]} - X_1\|^2\} \leq cn^{-2/\max\{d,2\}} \quad (n \in \mathbb{N}).$$

This lemma in its first part is stated for $d \geq 3$ in Györfi et al. [7], Lemma 6.4, and implies the second part according to [7], p. 95. For $d = 2$ (and then obviously also for $d = 1$) it immediately follows from Liitiäinen et al. [12], 3.2 (with reference to [11]) and [13], Theorem 3.2.
For our purpose the weaker bound $cn^{-1/(d+2)}$ would suffice.
Proof of Theorem 4.1 Choose $L \in [0, \infty)$ such that $|Y_i| \leq L$ and denote by l_n the number of cubes of the partition $\mathcal{P}_n$ that cover the bounded support of μ. It holds $l_n = O(h_n^{-d})$. $c_1,\ c_2, \ldots$ will be suitable constants. Set

$$W_{n,i} := (Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]}).$$

First, according to [7], p. 465, we note

$$\begin{aligned} &\left| \frac{\sum_{i=1}^n W_{n,i} 1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)} - \frac{\sum_{i=1}^n W_{n,i} 1_{A_n(x)}(X_i)}{n\mu(A_n(x))} \right| \\ \leq\ & 4L^2 \left| \frac{\sum_{i=1}^n 1_{A_n(x)}(X_i)}{n\mu(A_n(x))} - 1 \right|, \end{aligned} \tag{25}$$

further

$$\begin{aligned} &\boldsymbol{E} \int \left| \frac{\sum_{i=1}^n 1_{A_n(x)}(X_i) - n\mu(A_n(x))}{n\mu(A_n(x))} \right| \mu(dx) \\ \leq\ & \int \frac{\sqrt{\boldsymbol{Var}(\sum_{i=1}^n 1_{A_n(x)}(X_i))}}{n\mu(A_n(x))} \mu(dx) \\ \leq\ & \frac{1}{\sqrt{n}} \int \frac{1}{\sqrt{\mu(A_n(x))}} \mu(dx) \\ \leq\ & \frac{1}{\sqrt{n}} \sqrt{\int \frac{1}{\mu(A_n(x))} \mu(dx)} \\ \leq\ & \sqrt{l_n/n} \\ \leq\ & c_1 n^{-\frac{1}{2}} h_n^{-\frac{d}{2}}. \end{aligned} \tag{26}$$

In the second step we show

$$\int \left| \frac{\sum_{i=1}^n \boldsymbol{E}\{W_{n,i} 1_{A_n(x)}(X_i)\}}{n\mu(A_n(x))} - \sigma^2(x) \right| \mu(dx) \le c_2 \left(h_n + n^{-2/\max\{d,2\}} \right), \tag{27}$$

i.e.

$$\int \left| \frac{\boldsymbol{E}\{W_{n,1} 1_{A_n(x)}(X_1)\}}{\mu(A_n(x))} - \sigma^2(x) \right| \mu(dx) \le c_2 \left(h_n + n^{-2/\max\{d,2\}} \right). \tag{28}$$

According to Liitiäinen et al. [12], proof of Theorem 3, or [13], Appendix, via conditioning with respect to $X_1, \dots, X_n$, we have

$$\begin{aligned} & \boldsymbol{E}\{W_{n,1} 1_{A_n(x)}(X_1)\} \\ = & \boldsymbol{E}\{(Y_1 - m(X_1))^2 1_{A_n(x)}(X_1)\} \\ & + \boldsymbol{E}\left\{ \left(m(X_1) - m(X_{N[1,1]})\right) \left(m(X_1) - m(X_{N[1,2]})\right) 1_{A_n(x)}(X_1) \right\}. \end{aligned}$$

Then

$$\begin{aligned} & \int \left| \frac{\boldsymbol{E}\{(Y_1 - m(X_1))^2 1_{A_n(x)}(X_1)\}}{\mu(A_n(x))} \} - \sigma^2(x) \right| \mu(dx) \\ = & \int \left| \frac{\int \sigma^2(t) 1_{A_n(x)}(t) \mu(dt)}{\mu(A_n(x))} - \sigma^2(x) \right| \mu(dx) \\ \le & \int \frac{\int [\sigma^2(t) - \sigma^2(x)] 1_{A_n(x)}(t) \mu(dt)}{\mu(A_n(x))} \mu(dx) \\ \le & C \int \frac{\int \|t - x\| 1_{A_n(x)}(t) \mu(dt)}{\mu(A_n(x))} \mu(dx) \\ & \text{(by (23))} \\ \le & C\sqrt{d} h_n \int \frac{\int 1_{A_n(x)}(t) \mu(dt)}{\mu(A_n(x))} \mu(dx) \le C\sqrt{d} h_n. \end{aligned}$$

Further

$$\begin{aligned} & \int \left| \frac{\boldsymbol{E}\left\{ \left(m(X_1) - m(X_{N[1,1]})\right) \left(m(X_1) - m(X_{N[1,2]})\right) 1_{A_n(x)}(X_1) \right\}}{\mu(A_n(x))} \right| \mu(dx) \\ \le & \frac{1}{2} \int \frac{\boldsymbol{E}\left\{ \left| m(X_1) - m(X_{N[1,1]}) \right|^2 1_{A_n(x)}(X_1) \right\}}{\mu(A_n(x))} \mu(dx) \\ & + \frac{1}{2} \int \frac{\boldsymbol{E}\left\{ \left| m(X_1) - m(X_{N[1,2]}) \right|^2 1_{A_n(x)}(X_1) \right\}}{\mu(A_n(x))} \mu(dx) \\ \le & \frac{1}{2} D^2 \left[\boldsymbol{E}\left\{ \|X_{N[1,1]} - X_1\|^2 \right\} + \boldsymbol{E}\left\{ \|X_{N[1,2]} - X_1\|^2 \right\} \right] \\ & \text{(by (24) and } \int \left[1_{A_n(x)}(t) / \mu(A_n(x)) \right] \mu(dx) \le 1 \text{ for each } t \in \mathbb{R}^d) \\ \le & c_3 n^{-2/\max\{d,2\}} \end{aligned}$$

by Lemma 4.2. Thus (28) and (27) are obtained.
In the third step we show

$$\int \boldsymbol{E} \left| \frac{\sum_{i=1}^n \left[W_{n,i} 1_{A_n(x)}(X_i) - \boldsymbol{E}\{W_{n,i} 1_{A_n(x)}(X_i)\} \right]}{n\mu(A_n(x))} \right| \mu(dx) \le c_4 n^{-\frac{1}{2}} h_n^{-\frac{d}{2}}. \tag{29}$$

The left-hand side is bounded by

$$\int \frac{\sqrt{\boldsymbol{Var}\{\sum_{i=1}^n W_{n,i} 1_{A_n(x)}(X_i)\}}}{n\mu(A_n(x))} \mu(dx).$$

As in the proof of Theorem 2.5 we apply the Efron-Stein inequality (Lemma 2.6) and obtain, compare (18),

$$\boldsymbol{Var}\left\{\sum_{i=1}^{n} W_{n,i} 1_{A_n(x)}(X_i)\right\} \le c_5 n L^4 \boldsymbol{E}\{1_{A_n(x)}(X)\} = c_6 n \mu(A_n(x)).$$

Further

$$\int \frac{\sqrt{\mu(A_n(x))}}{\mu(A_n(x))}\mu(dx) \le \sqrt{\int \frac{1}{\mu(A_n(x))}\mu(dx)} \le c_7\sqrt{l_n} \le c_8 h_n^{-d/2}.$$

Thus (29) is obtained.
In the last step we gather (25), (26), (27), (29) and obtain

$$\begin{aligned} & \boldsymbol{E}\left\{\int |\sigma_n^2(x) - \sigma^2(x)|\mu(dx)\right\} \\ \le\ & c_9\left(n^{-\frac{1}{2}} h_n^{-\frac{d}{2}} + h_n + n^{-2/\max\{d,2\}}\right) \\ \le\ & c_{10} n^{-\frac{1}{d+2}} \end{aligned}$$

by the choice of (h_n). Thus the assertion is obtained. ∎

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