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José Carlos Díaz Ramos , Miguel Domínguez Vázquez, Andreas Kollross

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# POLAR ACTIONS ON COMPLEX HYPERBOLIC SPACES

ABSTRACT. We classify polar actions on complex hyperbolic spaces  $\mathbb{C}H^n$  up to orbit equivalence.

## 1. INTRODUCTION AND MAIN RESULT

A proper isometric Lie group action on a Riemannian manifold is called *polar* if there exists an immersed submanifold that meets every orbit orthogonally. Such a submanifold is then called a *section* of the action.

Let  $\mathbb{C}H^n = G/K$  be the complex hyperbolic  $n$ -space, where  $G = SU(1, n)$  and  $K = S(U(1)U(n))$  is the isotropy group of  $G$  at some point  $o$ . Consider the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  with respect to  $o$ . Choose a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  and let  $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$  be the root space decomposition with respect to  $\mathfrak{a}$ . Set  $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0 \cong \mathfrak{u}(n-1)$ . Since  $\mathfrak{k}_0$  acts on the root space  $\mathfrak{g}_\alpha$ , the center of  $\mathfrak{k}_0$  induces a natural complex structure  $J$  on  $\mathfrak{g}_\alpha$  which makes it isomorphic to  $\mathbb{C}^{n-1}$ . On the other hand, we call a subset of  $\mathfrak{g}_\alpha$  a *real subspace* of  $\mathfrak{g}_\alpha$  if it is a linear subspace of  $\mathfrak{g}_\alpha$ , where  $\mathfrak{g}_\alpha$  is viewed as a real vector space. Assume  $\mathfrak{g}_\alpha$  is endowed with the inner product given by the restriction of the Killing form of  $\mathfrak{g}$ . A real subspace  $\mathfrak{w}$  of  $\mathfrak{g}_\alpha$  is said to be *totally real* if  $\mathfrak{w} \perp J(\mathfrak{w})$ .

In this paper, we prove the following classification result:

**Theorem A.** *For each of the Lie algebras  $\mathfrak{h}$  below, the corresponding connected subgroup of  $U(1, n)$  acts polarly on  $\mathbb{C}H^n$ :*

- (i)  $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{so}(1, k) \subset \mathfrak{u}(n-k) \oplus \mathfrak{su}(1, k)$ ,  $k \in \{0, \dots, n\}$ , where  $\mathfrak{q}$  is a subalgebra of  $\mathfrak{u}(n-k)$  such that the corresponding subgroup  $Q$  of  $U(n-k)$  acts polarly with a totally real section on  $\mathbb{C}^{n-k}$ .
- (ii)  $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{su}(1, n)$ , where  $\mathfrak{b}$  is a linear subspace of  $\mathfrak{a}$ ,  $\mathfrak{w}$  is a real subspace of  $\mathfrak{g}_\alpha$ , and  $\mathfrak{q}$  is a subalgebra of  $\mathfrak{k}_0$  which normalizes  $\mathfrak{w}$  and such that the connected subgroup of  $SU(1, n)$  with Lie algebra  $\mathfrak{q}$  acts polarly with a totally real section on the orthogonal complement of  $\mathfrak{w}$  in  $\mathfrak{g}_\alpha$ .

*Conversely, every polar action on  $\mathbb{C}H^n$  is orbit equivalent to one of the actions above.*

In case (i) of Theorem A, one orbit of the  $H$ -action is a totally geodesic  $\mathbb{R}H^k$  and the other orbits are contained in the distance tubes around it. In case (ii), if  $\mathfrak{b} = \mathfrak{a}$ , one  $H$ -orbit of minimal orbit type contains a geodesic line, while if  $\mathfrak{b} = 0$ , any  $H$ -orbit of minimal orbit type is contained in a horosphere.

We would like to remark here that Theorem A actually provides many examples of polar actions on  $\mathbb{C}H^n$ . Indeed, for every choice of a real subspace  $\mathfrak{w}$  in  $\mathfrak{g}_\alpha$ , there is at least one polar action as described in part (ii) of Theorem A, see Section 3.

The motivation of our work can be tracked down to the work of Dadok [11] who classified polar representations on Euclidean spaces. Several years later, the interest of classifying polar and hyperpolar actions on symmetric spaces of compact type is stated in [15] (an isometric action is said to hyperpolar if it is polar and the section is flat). The classification of polar actions on compact symmetric spaces of rank one was obtained by Podestà and Thorbergsson [25]. This

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classification shows that there are examples of polar actions on symmetric spaces of rank one that are polar but not hyperpolar. Hyperpolar actions on irreducible symmetric spaces of compact type were classified by the third author in [17]. The lack of examples of polar actions that are not hyperpolar on irreducible symmetric spaces of compact type and higher rank, led Biliotti [10] to formulate the following conjecture: a polar action on an irreducible symmetric space of compact type and higher rank is hyperpolar. The third author answered this question in the affirmative for symmetric spaces with simple isometry group [18], and for the exceptional simple Lie groups [19]. The final step was given by Kollross and Lytchak [21] who showed that Biliotti's conjecture can be answered in the affirmative: a polar action on an irreducible symmetric space of compact type and rank higher than one is hyperpolar, and hence, the classification follows from [17]. It is worthwhile to mention that the classification of polar actions on reducible symmetric spaces cannot be obtained from the corresponding classification in irreducible ones.

While there has been certain progress in the study of polar actions on compact symmetric spaces, the situation in the noncompact case remains largely open. Wu [30] classified polar actions on real hyperbolic spaces and showed that, up to orbit equivalence, they are products of a noncompact factor, which is either the isometry group of a lower dimensional real hyperbolic space or the nilpotent part of its Iwasawa decomposition, and a compact factor, which comes from the isotropy representation of a symmetric space. In particular, there are finitely many examples of polar actions on a real hyperbolic space up to orbit equivalence. Berndt and the first author obtained in [5] the classification of polar actions on the complex hyperbolic plane  $\mathbb{C}H^2$ , showing that there are exactly nine examples up to orbit equivalence. No other classification of polar actions was known on a symmetric space of noncompact type. The aim of this paper is precisely to present the classification of polar actions on complex hyperbolic spaces of any dimension.

An important fact to bear in mind here is that, in general, duality cannot be applied to derive classifications of polar actions in noncompact symmetric spaces from the corresponding classifications in the compact setting. A quick way to see this is the following. It was proved in [25] that polar actions on irreducible symmetric spaces of compact type always have singular orbits; however, a horosphere foliation on a real hyperbolic space is polar but does not have singular orbits, so it cannot be obtained from duality. Nevertheless, there are certain situations where duality can be used to obtain partial classifications. The first and the third author derived in [13] the classification of polar actions with a fixed point on symmetric spaces using this method. Remarkably, it can be shown that a polar action with a fixed point in a reducible symmetric space splits as a product of polar actions on each factor. The third author explored this idea a bit further and obtained a classification of polar actions by algebraic reductive subgroups using duality in [20].

Berndt and Tamaru [8] classified cohomogeneity one actions on complex hyperbolic spaces, the quaternionic hyperbolic plane, and the Cayley hyperbolic plane. Note that in rank one an isometric action is hyperpolar if and only if it is of cohomogeneity one. The classification remains open in quaternionic hyperbolic spaces  $\mathbb{H}H^n$ ,  $n \geq 3$ , and in symmetric spaces of higher rank. See [9] for more information on cohomogeneity one actions on symmetric spaces of noncompact type. As we mentioned earlier, a polar action on a symmetric space of compact type always has singular orbits. Motivated by this fact Berndt, Tamaru and the first author studied hyperpolar actions on symmetric spaces that have no singular orbits [7] and obtained a complete classification. It was also shown in this paper that there are polar actions on symmetric spaces of noncompact type and rank higher than one that are not hyperpolar unlike in the compact setting. This classification can be improved in complex hyperbolic spaces, where Berndt and the first author classified polar homogeneous foliations [6]. The main result of this paper contains [6] and [8] as particular cases.

This paper is organized as follows. In Section 2 we review the basic facts and notations on complex hyperbolic spaces (§2.1), polar actions (§2.2), and real vector subspaces of complex vector spaces (§2.3). The results of Subsection 2.3 will be crucial for the rest of the paper. Section 3 is devoted to present the new examples that appear in Theorem A. We also present here an outline of the proof of Theorem A. This proof has two main parts depending on whether the group acting

upon leaves a totally geodesic subspace invariant (Section 4) or is contained in a maximal parabolic subgroup of  $SU(1, n)$  (Section 5). We conclude in Section 6 with the proof of Theorem A.

## 2. PRELIMINARIES

In this section we introduce the main known results and notation used throughout this paper. We would like to emphasize the importance of Subsection 2.3, which is pivotal in the construction and classification of new examples of polar actions on complex hyperbolic spaces.

As a matter of notation, if  $U_1$  and  $U_2$  are two linear subspaces of a vector space  $V$ , then  $U_1 \oplus U_2$  denotes their (not necessarily orthogonal) direct sum. We will frequently use the following notation for the orthogonal complement of a subspace of a real vector space endowed with a scalar product, namely, by  $V \ominus U$  we denote the orthogonal complement of the linear subspace  $U$  in the Euclidean vector space  $V$ .

**2.1. The complex hyperbolic space.** In this subsection we recall some well-known facts and notation on the structure of the complex hyperbolic space as a symmetric space. This will be fundamental for the rest of the work. As usual, Lie algebras are written in gothic letters.

We will denote by  $\mathbb{C}H^n$  the complex hyperbolic space with constant holomorphic sectional curvature  $-1$ . As a symmetric space,  $\mathbb{C}H^n$  is the coset space  $G/K$ , where  $G = SU(1, n)$ , and  $K = S(U(1)U(n))$  is the isotropy group at some point  $o \in \mathbb{C}H^n$ . Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition of  $\mathfrak{g}$  with respect to  $o$ , where  $\mathfrak{p}$  is the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$  with respect to the Killing form  $B$  of  $\mathfrak{g}$ . Denote by  $\theta$  the corresponding Cartan involution, which satisfies  $\theta|_{\mathfrak{k}} = \text{id}$  and  $\theta|_{\mathfrak{p}} = -\text{id}$ . Note that the orthogonal projections onto  $\mathfrak{k}$  and  $\mathfrak{p}$  are  $\frac{1}{2}(1 + \theta)$  and  $\frac{1}{2}(1 - \theta)$ , respectively. Let  $\text{ad}$  and  $\text{Ad}$  be the adjoint maps of  $\mathfrak{g}$  and  $G$ , respectively. It turns out that  $\langle X, Y \rangle = -B(\theta X, Y)$  defines a positive definite inner product on  $\mathfrak{g}$  satisfying the relation  $\langle \text{ad}(X)Y, Z \rangle = -\langle Y, \text{ad}(\theta X)Y \rangle$  for all  $X, Y, Z \in \mathfrak{g}$ . Moreover, we can identify  $\mathfrak{p}$  with the tangent space  $T_o \mathbb{C}H^n$  of  $\mathbb{C}H^n$  at the point  $o$ .

Since  $\mathbb{C}H^n$  has rank one, any maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  is 1-dimensional. For each linear functional  $\lambda$  on  $\mathfrak{a}$ , define  $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \text{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$ . Then  $\mathfrak{a}$  induces the restricted root space decomposition  $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , which is an orthogonal direct sum with respect to  $\langle \cdot, \cdot \rangle$  satisfying  $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \mathfrak{g}_{\lambda+\mu}$  and  $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ . Moreover,  $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$ , where  $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k} \cong \mathfrak{u}(n-1)$  is the normalizer of  $\mathfrak{a}$  in  $\mathfrak{k}$ . The root space  $\mathfrak{g}_\alpha$  has dimension  $2n-2$ , while  $\mathfrak{g}_{2\alpha}$  is 1-dimensional, and both are normalized by  $\mathfrak{k}_0$ .

Let us define  $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , which is a nilpotent subalgebra of  $\mathfrak{g}$  isomorphic to the  $(2n-1)$ -dimensional Heisenberg algebra. The corresponding Iwasawa decomposition of  $\mathfrak{g}$  is  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ . The connected subgroup of  $G$  with Lie algebra  $\mathfrak{a} \oplus \mathfrak{n}$  acts simply transitively on  $\mathbb{C}H^n$ . One may endow  $AN$ , and then  $\mathfrak{a} \oplus \mathfrak{n}$ , with the left-invariant metric  $\langle \cdot, \cdot \rangle_{AN}$  and the complex structure  $J$  that make  $\mathbb{C}H^n$  and  $AN$  isometric and isomorphic as Kähler manifolds. Then  $\langle X, Y \rangle_{AN} = \langle X_\mathfrak{a}, Y_\mathfrak{a} \rangle + \frac{1}{2} \langle X_\mathfrak{n}, Y_\mathfrak{n} \rangle$  for  $X, Y \in \mathfrak{a} \oplus \mathfrak{n}$ ; here subscripts mean orthogonal projection. The complex structure  $J$  on  $\mathfrak{a} \oplus \mathfrak{n}$  leaves  $\mathfrak{g}_\alpha$  invariant, turning  $\mathfrak{g}_\alpha$  into an  $(n-1)$ -dimensional complex vector space  $\mathbb{C}^{n-1}$ . Moreover,  $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$ .

Let  $B \in \mathfrak{a}$  be a unit vector and define  $Z = JB \in \mathfrak{g}_{2\alpha}$ . Then  $\langle B, B \rangle = \langle B, B \rangle_{AN} = 1$  and  $\langle Z, Z \rangle = 2\langle Z, Z \rangle_{AN} = 2$ . The Lie bracket of  $\mathfrak{a} \oplus \mathfrak{n}$  is given by

$$[aB + U + xZ, bB + V + yZ] = -\frac{b}{2}U + \frac{a}{2}V + \left(-bx + ay + \frac{1}{2}\langle JU, V \rangle\right)Z,$$

where  $a, b, x, y \in \mathbb{R}$ , and  $U, V \in \mathfrak{g}_\alpha$ . Let us also define  $\mathfrak{p}_\lambda = (1 - \theta)\mathfrak{g}_\lambda$ , the projection onto  $\mathfrak{p}$  of the restricted root spaces. Then  $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ . If the complex structure on  $\mathfrak{p}$  is denoted by  $i$ , then we have that  $2iB = (1 - \theta)Z$ , and  $i(1 - \theta)U = (1 - \theta)JU$  for every  $U \in \mathfrak{g}_\alpha$ .

We state now two lemmas that will be used frequently along the article.

**Lemma 2.1.** *We have:*

- (a)  $[\theta X, Z] = -JX$  for each  $X \in \mathfrak{g}_\alpha$ .

(b)  $\langle T, (1 + \theta)[\theta X, Y] \rangle = 2\langle [T, X], Y \rangle$ , for any  $X, Y \in \mathfrak{g}_\alpha$  and  $T \in \mathfrak{k}_0$ .

*Proof.* See [6, Lemma 2.1].  $\square$

**Lemma 2.2.** *The orthogonal projection map  $\frac{1}{2}(1 - \theta): \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \rightarrow \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$  defines an equivalence between the adjoint  $K_0$ -representation on  $\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$  and the adjoint  $K_0$ -representation on  $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ . Moreover, this equivalence is an isometry between  $(\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}, \langle \cdot, \cdot \rangle_{AN})$  and  $(\mathfrak{p}, \langle \cdot, \cdot \rangle)$ , and  $\frac{1}{2}(1 - \theta): \mathfrak{g}_\alpha \rightarrow \mathfrak{p}_\alpha$  is a complex linear map.*

*Proof.* The first part follows from the fact that  $\theta$  is a  $K$ -equivariant, hence  $K_0$ -equivariant, map on  $\mathfrak{g}$ . The other claims follow from the facts stated above in this subsection.  $\square$

**2.2. Polar actions.** Let  $M$  be a Riemannian manifold and  $I(M)$  its isometry group. It is known that  $I(M)$  is a Lie group. Let  $H$  be a connected closed subgroup of  $I(M)$ . The action of  $H$  on  $M$  is called *polar* if there exists an immersed submanifold  $\Sigma$  of  $M$  such that:

- (1)  $\Sigma$  intersects all the orbits of the  $H$ -action, and
- (2) for each  $p \in \Sigma$ , the tangent space of  $\Sigma$  at  $p$ ,  $T_p\Sigma$ , and the tangent space of the orbit through  $p$  at  $p$ ,  $T_p(H \cdot p)$ , are orthogonal.

In such a case, the submanifold  $\Sigma$  is called a *section* of the  $H$ -action. The action of  $H$  is called *hyperpolar* if the section  $\Sigma$  is flat in its induced Riemannian metric.

Two isometric Lie group actions on two Riemannian manifolds  $M$  and  $N$  are said to be *orbit equivalent* if there is an isometry  $M \rightarrow N$  which maps connected components of orbits onto connected components of orbits. They are said to be *conjugate* if there exists an equivariant isometry  $M \rightarrow N$ .

The final aim of our research is to classify polar actions on a given Riemannian manifold up to orbit equivalence. In this paper we accomplish this task for complex hyperbolic spaces. See the survey articles [27] and [28] for more information and references on polar actions.

Since  $\mathbb{C}H^n$  is of rank one, a polar action on  $\mathbb{C}H^n$  is hyperpolar if and only if it is of cohomogeneity one, i.e. the orbits of maximal dimension are hypersurfaces. Conversely, any action of cohomogeneity one on  $\mathbb{C}H^n$  (or any other Riemannian symmetric space) is hyperpolar. Cohomogeneity one actions on complex hyperbolic spaces have been classified by Berndt and Tamaru in [8].

From now on we focus on polar actions on complex hyperbolic spaces and recall or prove some facts that will be used later in this article. We begin with a criterion that allows us to decide whether an action is polar or not. The first such criterion of polarity is credited to Gorodski [14].

**Proposition 2.3.** *Let  $M = G/K$  be a Riemannian symmetric space of noncompact type, and let  $\Sigma$  be a connected totally geodesic submanifold of  $M$  with  $o \in \Sigma$ . Let  $H$  be a closed subgroup of  $I(M)$ . Then  $H$  acts polarly on  $M$  with section  $\Sigma$  if and only if  $T_o\Sigma$  is a section of the slice representation of  $H_o$  on  $\nu_o(H \cdot o)$ , and  $\langle \mathfrak{h}, T_o\Sigma \oplus [T_o\Sigma, T_o\Sigma] \rangle = 0$ .*

*In this case, the following conditions are satisfied:*

- (a)  $T_o\Sigma \oplus [\mathfrak{h}_o, \xi] = \nu_o(H \cdot o)$  for each regular normal vector  $\xi \in \nu_o(H \cdot o)$ .
- (b)  $T_o\Sigma \oplus [\mathfrak{h}_o, T_o\Sigma] = \nu_o(H \cdot o)$ .
- (c)  $\text{Ad}(H_o)T_o\Sigma = \nu_o(H \cdot o)$ .

*Proof.* Follows from [5, Corollary 3.2] and from well-known facts on polar representations of compact groups [11].  $\square$

If  $N$  is a submanifold of  $\mathbb{C}H^n$ , then  $N$  is said to be *totally real* if for each  $p \in N$  the tangent space  $T_pN$  is a totally real subspace of  $T_p\mathbb{C}H^n$ , that is,  $JT_pN$  is orthogonal to  $T_pN$ . See §2.3 for more information of totally real subspaces of complex vector spaces. The next theorem shows that sections are necessarily totally real.

**Proposition 2.4.** *Let  $H$  act nontrivially, nontransitively, and polarly on the complex hyperbolic space  $\mathbb{C}H^n$ , and let  $\Sigma$  be a section of this action. Then,  $\Sigma$  is a totally real submanifold of  $\mathbb{C}H^n$ .*



*Proof.* Since the action of  $H$  is polar, the section  $\Sigma$  is a totally geodesic submanifold of  $\mathbb{C}H^n$ , hence  $\Sigma$  is either totally real or complex. Assume that  $\Sigma$  is complex.

Since all sections are of the form  $h(\Sigma)$ , with  $h \in H$ , and the isometries of  $H$  are holomorphic, it follows that any principal orbit is almost complex. It is a well-known fact that an almost complex submanifold in a Kähler manifold is Kähler. Since every  $H$ -equivariant normal vector field on a principal orbit is parallel with respect to the normal connection [3, Corollary 3.2.5], then this principal orbit is either a point or  $\mathbb{C}H^n$  (see for example [1]), contradiction. Therefore  $\Sigma$  is totally real.  $\square$

**2.3. The structure of a real subspace of a complex vector space.** Let us denote by  $J$  the complex structure of the complex vector space  $\mathbb{C}^n$ . We view  $\mathbb{C}^n$  as a Euclidean vector space with the scalar product given by the real part of the standard Hermitian scalar product. We define a *real subspace* of  $\mathbb{C}^n$  to be an  $\mathbb{R}$ -linear subspace of the real vector space obtained from  $\mathbb{C}^n$  by restricting the scalars to the real numbers. Let  $V$  be a real subspace of  $\mathbb{C}^n$ . We will denote by  $\pi_V$  the orthogonal projection map onto  $V$ .

The *Kähler angle* of a nonzero vector  $v \in V$  with respect to  $V$  is defined to be the angle between  $Jv$  and  $V$  or, equivalently, the value  $\varphi \in [0, \pi/2]$  such that  $\langle \pi_V Jv, \pi_V Jv \rangle = \cos^2(\varphi) \langle v, v \rangle$ . We say that  $V$  has *constant Kähler angle*  $\varphi$  if the Kähler angle of every nonzero vector  $v \in V$  with respect to  $V$  is  $\varphi$ . In particular,  $V$  is a *complex subspace* if and only if it has constant Kähler angle 0; it is a *totally real subspace* if and only if it has constant Kähler angle  $\pi/2$ .

*Example 2.5.* If  $\{e_1, \dots, e_n\}$  and  $\{f_1, \dots, f_n\}$  both are orthonormal bases of  $\mathbb{C}^n$ , then the real subspace  $V_\varphi$  of  $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$  generated by

$$\left\{ \cos\left(\frac{\varphi}{2}\right)e_1 + \sin\left(\frac{\varphi}{2}\right)Jf_1, \cos\left(\frac{\varphi}{2}\right)Je_1 + \sin\left(\frac{\varphi}{2}\right)f_1, \dots, \cos\left(\frac{\varphi}{2}\right)e_n + \sin\left(\frac{\varphi}{2}\right)Jf_n, \cos\left(\frac{\varphi}{2}\right)Je_n + \sin\left(\frac{\varphi}{2}\right)f_n \right\}$$

has constant Kähler angle  $\varphi \in [0, \pi/2)$ . Conversely, any subspace of constant Kähler angle  $\varphi \in [0, \pi/2)$  and dimension  $2n$  of  $\mathbb{C}^{2n}$  can be constructed in this way, see [2].

For general real subspaces of a complex vector space, we have the following structure result.

**Theorem 2.6.** *Let  $V$  be any real subspace of  $\mathbb{C}^n$ . Then  $V$  can be decomposed in a unique way as an orthogonal sum of subspaces  $V_i$ ,  $i = 1, \dots, r$ , such that:*

- (a) *Each real subspace  $V_i$  of  $\mathbb{C}^n$  has constant Kähler angle  $\varphi_i$ .*
- (b)  *$\mathbb{C}V_i \perp \mathbb{C}V_j$ , for every  $i \neq j$ ,  $i, j \in \{1, \dots, r\}$ .*
- (c)  *$\varphi_1 < \varphi_2 < \dots < \varphi_r$ .*

*Proof.* The endomorphism  $P = \pi_V \circ J$  of  $V$  is clearly skew-symmetric, i.e.  $\langle Pv, w \rangle = -\langle v, Pw \rangle$  for every  $v, w \in V$ . Then, there exists an orthonormal basis of  $V$  for which  $P$  takes a block diagonal form with  $2 \times 2$  skew-symmetric matrix blocks, and maybe one zero matrix block. Since  $P$  is skew-symmetric, its nonzero eigenvalues are imaginary. Assume then that the distinct eigenvalues of  $P$  are  $\pm i\lambda_1, \dots, \pm i\lambda_r$  (maybe one of them is zero). We can and will further assume that  $|\lambda_1| > \dots > |\lambda_r|$ .

Now consider the quadratic form  $\Psi: V \rightarrow \mathbb{R}$  defined by  $\Psi(v) = \langle Pv, Pv \rangle = -\langle P^2v, v \rangle$  for  $v \in V$ . The matrix of this quadratic form  $\Psi$  (or of the endomorphism  $-P^2$ ) with respect to the basis fixed above is diagonal with entries  $\lambda_1^2, \dots, \lambda_r^2$ . For each  $i = 1, \dots, r$ , let  $V_i$  be the eigenspace of  $-P^2$  corresponding the eigenvalue  $\lambda_i^2$ . Let  $v \in V_i$  be a unit vector. Then

$$\langle \pi_{V_i} Jv, \pi_{V_i} Jv \rangle = \langle Pv, \pi_{V_i} Jv \rangle = \langle Pv, Pv \rangle = \Psi(v) = \lambda_i^2,$$

where in the second and last equalities we have used that  $Pv \in V_i$ . This means that each subspace  $V_i$  has constant Kähler angle  $\varphi_i$ , where  $\varphi_i$  is the unique value in  $[0, \frac{\pi}{2}]$  such that  $\lambda_i^2 = \cos^2(\varphi_i)$ .

By construction, it is clear that  $V_i \perp V_j$  and  $JV_i \perp JV_j$  for  $i \neq j$ . Since for every  $v \in V_i$  and  $w \in V_j$ ,  $i \neq j$ , we have that  $\langle Jv, w \rangle = \langle Pv, w \rangle = 0$ , we also get that  $JV_i \perp V_j$  if  $i \neq j$ . Hence  $\mathbb{C}V_i \perp \mathbb{C}V_j$  if  $i \neq j$ .

Property (c) follows from the assumption that  $|\lambda_1| > \dots > |\lambda_r|$ , and this also implies the uniqueness of the decomposition.  $\square$

It is convenient to change the notation of Theorem 2.6 slightly. Let  $V$  be any real subspace of  $\mathbb{C}^n$ , and let  $V = \bigoplus_{\varphi \in \Phi} V_\varphi$  be the decomposition stated in Theorem 2.6, where  $V_\varphi$  has constant Kähler angle  $\varphi \in [0, \pi/2]$ , and  $\Phi$  is the set of all possible Kähler angles arising in this decomposition. Note that according to Theorem 2.6, this decomposition is unique up to the order of the factors. We agree to write  $V_\varphi = 0$  if  $\varphi \notin \Phi$ . The subspaces  $V_0$  and  $V_{\pi/2}$  (which can be zero) play a somewhat distinguished role in the calculations that follow, so we will denote  $\Phi^* = \{\varphi \in \Phi : \varphi \neq 0, \pi/2\}$ . Then, the above decomposition is written as

$$V = V_0 \oplus \left( \bigoplus_{\varphi \in \Phi^*} V_\varphi \right) \oplus V_{\pi/2}.$$

For each  $\varphi \in \Phi^* \cup \{0\}$ , we define  $J_\varphi: V_\varphi \rightarrow V_\varphi$  by  $J_\varphi = \frac{1}{\cos(\varphi)}(\pi_{V_\varphi} \circ J)$ . This is clearly a skew-symmetric and orthogonal endomorphism of  $V_\varphi$  (see the proof of Theorem 2.6). Therefore  $(V_\varphi, J_\varphi)$  is a complex vector space for every  $\varphi \in \Phi^* \cup \{0\}$ . Note that  $J_0 = J|_{V_0}$ . Let  $U(V_\varphi)$  be the group of all unitary transformations of the complex vector space  $(V_\varphi, J_\varphi)$ .

**Lemma 2.7.** *Let  $V$  be a real subspace of constant Kähler angle  $\varphi \neq 0$  in  $\mathbb{C}^n$ . Then the real subspace  $\mathbb{C}V \ominus V$  of  $\mathbb{C}^n$  has the same dimension as  $V$  and constant Kähler angle  $\varphi$ .*

*Proof.* See for example [4, page 135].  $\square$

Let  $V^\perp = \mathbb{C}^n \ominus V$ , where as usual  $\ominus$  denotes the orthogonal complement. Then, Lemma 2.7 implies that the decomposition stated in Theorem 2.6 can be written as

$$V^\perp = V_0^\perp \oplus \left( \bigoplus_{\varphi \in \Phi^*} V_\varphi^\perp \right) \oplus V_{\pi/2}^\perp, \quad \text{where } \mathbb{C}V_\varphi = V_\varphi \oplus V_\varphi^\perp \text{ for each } \varphi \in \Phi^* \cup \{\pi/2\}.$$

We define  $m_\varphi = \dim V_\varphi$  and  $m_\varphi^\perp = \dim V_\varphi^\perp$ . For every  $\varphi \neq 0$  we have  $m_\varphi = m_\varphi^\perp$  by Lemma 2.7, but  $V_0$  and  $V_0^\perp$  are both complex subspaces of  $\mathbb{C}^n$ , possibly of different dimension.

**Lemma 2.8.** *Let  $V$  be a real subspace of  $\mathbb{C}^n$ . Let  $U(n)_V$  be the subgroup of  $U(n)$  consisting of all the elements  $A \in U(n)$  such that  $AV = V$ . Then, we have the canonical isomorphism*

$$U(n)_V \cong \left[ \prod_{\varphi \in \Phi^* \cup \{0\}} U(V_\varphi) \right] \times O(V_{\pi/2}) \times U(V_0^\perp).$$

where we assume that  $V_\varphi$ ,  $\varphi \in \Phi^* \cup \{0\}$ , is endowed with the complex structure given by  $J_\varphi = \frac{1}{\cos(\varphi)}(\pi_{V_\varphi} \circ J)$ , and that  $V_0^\perp$  is endowed with the complex structure given by the restriction of  $J$ .

*Proof.* Let  $A \in U(n)$  be such that  $AV = V$ . Then  $A$  commutes with  $J$  and  $\pi_V$  and hence leaves the eigenspaces of  $-P^2$  invariant (see the proof of Theorem 2.6). Thus  $AV_\varphi = V_\varphi$ . Since we also have  $AV^\perp = V^\perp$ , it follows that  $AV_\varphi^\perp = V_\varphi^\perp$ .

Let  $\varphi \in \Phi \cup \{0\}$ . Since  $AV_\varphi = V_\varphi$  and  $AV_\varphi^\perp = V_\varphi^\perp$  we have  $ACV_\varphi = \mathbb{C}V_\varphi$ . Clearly,  $A \circ \pi_{V_\varphi}|_{V_\varphi} = \pi_{V_\varphi} \circ A|_{V_\varphi}$ , and  $A \circ \pi_{V_\varphi}|_{\mathbb{C}^n \ominus V_\varphi} = 0 = \pi_{V_\varphi} \circ A|_{\mathbb{C}^n \ominus V_\varphi}$ . Hence,  $A \circ \pi_{V_\varphi} = \pi_{V_\varphi} \circ A$ . Since  $AJ = JA$  as well, we have that  $A \circ J_\varphi|_{V_\varphi} = J_\varphi \circ A|_{V_\varphi}$  on  $V_\varphi$ , and thus,  $A|_{V_\varphi} \in U(V_\varphi)$ . If  $\varphi = \pi/2$  then we have  $AV_{\pi/2} = V_{\pi/2}$ , and clearly,  $A|_{V_{\pi/2}}$  is an orthogonal transformation of  $V_{\pi/2}$ . Moreover, we have  $A|_{V_0^\perp} \in U(V_0^\perp)$ . We define a map

$$F: U(n)_V \rightarrow \left[ \prod_{\varphi \in \Phi^* \cup \{0\}} U(V_\varphi) \right] \times O(V_{\pi/2}) \times U(V_0^\perp)$$

by requiring that the projection onto each factor is given by the corresponding restriction, that is, the  $U(V_\varphi)$ -projection of  $F(A)$  is given by  $A|_{V_\varphi}$ , the  $O(V_{\pi/2})$ -projection of  $F(A)$  is  $A|_{V_{\pi/2}}$ , and the  $U(V_0^\perp)$ -projection of  $F(A)$  is  $A|_{V_0^\perp}$ .

Since every element in  $U(n)_V$  leaves the subspaces  $V_\varphi$ ,  $\varphi \in \Phi$ , and  $V_0^\perp$  invariant, the map thus defined is a homomorphism. Let us show injectivity and surjectivity. Let  $A_\varphi \in U(V_\varphi)$  for each  $\varphi \in \Phi^* \cup \{0\}$ , let  $A_{\pi/2} \in O(V_{\pi/2})$ , and let  $A_0^\perp \in U(V_0^\perp)$ . If  $A \in U(n)_V$  and  $v \in JV_\varphi$  for  $\varphi \in \Phi$ , then  $Av$  is determined by  $A_\varphi$  and  $v$ , since  $Av = -AJ^2v = -JAJv = -JA_\varphi(Jv)$ . Since we have the direct sum decomposition

$$\mathbb{C}^n = \left[ \bigoplus_{\varphi \in \Phi} \mathbb{C}V_\varphi \right] \oplus V_0^\perp,$$

it follows that the unitary map  $A$  on  $\mathbb{C}^n$  is uniquely determined by the maps  $A_\varphi$ ,  $\varphi \in \Phi$ , and  $A_0^\perp$ . This shows injectivity.

Conversely, let  $A \in \left[ \prod_{\varphi \in \Phi^* \cup \{0\}} U(V_\varphi) \right] \times O(V_{\pi/2}) \times U(V_0^\perp)$ , and denote by  $A_\varphi$  the  $U(V_\varphi)$ -projection, by  $A_{\pi/2}$  the  $O(V_{\pi/2})$ -projection, and by  $A_0^\perp$  the  $U(V_0^\perp)$ -projection. Then, we may construct a map  $A \in U(n)_V$  by defining  $A(v + Jw) = A_\varphi v + JA_\varphi w$  for all  $v, w \in V_\varphi$ ,  $\varphi \in \Phi$ ,  $Av = A_0^\perp v$  for  $v \in V_0^\perp$ , and extending linearly. For the map  $A$  thus defined we have  $A|_{V_\varphi} = A_\varphi$  for  $\varphi \in \Phi$ , and  $A|_{V_0^\perp} = A_0^\perp$ . This proves surjectivity.  $\square$

### 3. NEW EXAMPLES OF POLAR ACTIONS

We will now construct new examples of polar actions on complex hyperbolic spaces. We will use the notation from Subsection 2.1.

Recall that the root space  $\mathfrak{g}_\alpha$  is a complex vector space, which we will identify with  $\mathbb{C}^{n-1}$ . Let  $\mathfrak{w}$  be a real subspace of  $\mathfrak{g}_\alpha$  and

$$\mathfrak{w} = \bigoplus_{\varphi \in \Phi} \mathfrak{w}_\varphi = \mathfrak{w}_0 \oplus \left( \bigoplus_{\varphi \in \Phi^*} \mathfrak{w}_\varphi \right) \oplus \mathfrak{w}_{\pi/2}$$

its decomposition as in Theorem 2.6, where  $\Phi$  is the set of all possible Kähler angles of vectors in  $\mathfrak{w}$ ,  $\Phi^* = \{\varphi \in \Phi : \varphi \neq 0, \pi/2\}$ , and  $\mathfrak{w}_\varphi$  has constant Kähler angle  $\varphi \in [0, \pi/2]$ . Similarly, define  $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$  and let

$$\mathfrak{w}^\perp = \mathfrak{w}_0^\perp \oplus \left( \bigoplus_{\varphi \in \Phi^*} \mathfrak{w}_\varphi^\perp \right) \oplus \mathfrak{w}_{\pi/2}^\perp$$

be the corresponding decomposition as in Theorem 2.6. We define  $m_\varphi = \dim \mathfrak{w}_\varphi$  and  $m_\varphi^\perp = \dim \mathfrak{w}_\varphi^\perp$ , and recall that  $m_\varphi = m_\varphi^\perp$  if  $\varphi \in (0, \pi/2]$ . Recall also that  $K_0$ , the connected subgroup of  $G = SU(1, n)$  with Lie algebra  $\mathfrak{k}_0$ , is isomorphic to  $U(n-1)$  and acts on  $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$  in the standard way. We know from Lemma 2.8 that the normalizer  $N_{K_0}(\mathfrak{w})$  of  $\mathfrak{w}$  in  $K_0$  has the form

$$(1) \quad N_{K_0}(\mathfrak{w}) \cong \left[ \prod_{\varphi \in \Phi^* \cup \{0\}} U(\mathfrak{w}_\varphi) \right] \times O(\mathfrak{w}_{\pi/2}) \times U(\mathfrak{w}_0^\perp).$$

This group leaves invariant each  $\mathfrak{w}_\varphi$  and each  $\mathfrak{w}_\varphi^\perp$ , and acts transitively on the unit sphere of these subspaces of constant Kähler angle. Moreover, it acts polarly on  $\mathfrak{w}^\perp$ , see Remark 3.2 below.

The following result provides a large family of new examples of polar actions on  $\mathbb{C}H^n$ .

**Theorem 3.1.** *Let  $\mathfrak{w}$  be a real subspace of  $\mathfrak{g}_\alpha$  and  $\mathfrak{b}$  a subspace of  $\mathfrak{a}$ . Let  $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , where  $\mathfrak{q}$  is any Lie subalgebra of  $\mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$  such that the corresponding connected subgroup  $Q$  of  $K$  acts polarly on  $\mathfrak{w}^\perp$  with section  $\mathfrak{s}$ . Assume  $\mathfrak{s}$  is a totally real subspace of  $\mathfrak{g}_\alpha$ . Then the connected subgroup  $H$  of  $G$  with Lie algebra  $\mathfrak{h}$  acts polarly on  $\mathbb{C}H^n$  with section  $\Sigma = \exp_o((\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)\mathfrak{s})$ .*

*Proof.* We have that  $T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)\mathfrak{s}$  and  $\nu_o(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)\mathfrak{w}^\perp$ . Since  $\mathfrak{s} \subset \mathfrak{w}^\perp$ , it follows that  $T_o\Sigma \subset \nu_o(H \cdot o)$ . The slice representation of  $H_o$  on  $\nu_o(H \cdot o)$  leaves the subspaces  $\mathfrak{a} \ominus \mathfrak{b}$  and  $(1 - \theta)\mathfrak{w}^\perp$  invariant. For the first one the action is trivial, while for the second one the action is equivalent to the representation of  $Q$  on  $\mathfrak{w}^\perp$  (see Lemma 2.2), which is polar with section  $\mathfrak{s}$ . Hence, the slice representation of  $H_o$  on  $\nu_o(H \cdot o)$  is polar and  $T_o\Sigma$  is a section of it. Let  $v, w \in \mathfrak{s} \subset \mathfrak{w}^\perp$ . We have:

$$[(1 - \theta)v, (1 - \theta)w] = (1 + \theta)[v, w] - (1 + \theta)[\theta v, w] = -(1 + \theta)[\theta v, w].$$

The last equality holds because  $v$  and  $w$  lie in  $\mathfrak{s}$ , which is a totally real subspace of  $\mathfrak{g}_\alpha$ , and then  $[v, w] = \frac{1}{2}\langle Jv, w \rangle Z = 0$ . Since  $v, w \in \mathfrak{g}_\alpha$ , then  $\theta v \in \mathfrak{g}_{-\alpha}$  and  $[\theta v, w] \in \mathfrak{g}_0$ . Hence  $-(1 + \theta)[\theta v, w] \in \mathfrak{k}_0$ . Let  $X = T + aB + U + xZ \in \mathfrak{h}$ , where  $T \in \mathfrak{q}$ ,  $U \in \mathfrak{w}$  and  $a, x \in \mathbb{R}$ . Since  $\mathfrak{k}_0$  is orthogonal to  $\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , we have:

$$\langle [(1 - \theta)v, (1 - \theta)w], X \rangle = -\langle (1 + \theta)[\theta v, w], T \rangle = -2\langle [T, v], w \rangle = -4\langle [T, v], w \rangle_{AN} = 0,$$

where in the last equality we have used that the action of  $Q$  on  $\mathfrak{w}^\perp$  is a polar representation with section  $\mathfrak{s}$ . If  $\mathfrak{b} = \mathfrak{a}$ , the result then follows using the criterion in Proposition 2.3.

If  $\mathfrak{b} \neq \mathfrak{a}$  then  $\mathfrak{b} = 0$ . In this case, let  $v \in \mathfrak{s}$  and  $X = T + U + xZ \in \mathfrak{h}$ , where  $T \in \mathfrak{q}$ ,  $U \in \mathfrak{w}$ ,  $x \in \mathbb{R}$ . Then:

$$\langle [B, (1 - \theta)v], X \rangle = \langle (1 + \theta)[B, v], X \rangle = \frac{1}{2}\langle (1 + \theta)v, U \rangle = 0.$$

Since  $[B, B] = 0$ , by linearity and the skew-symmetry of the Lie bracket, it follows that  $\langle [T_o\Sigma, T_o\Sigma], \mathfrak{h} \rangle = 0$ . Again by Proposition 2.3, the result follows also in case  $\mathfrak{b} \neq \mathfrak{a}$ .  $\square$

*Remark 3.2.* In the special case  $Q = N_{K_0}(\mathfrak{w})$ , we obtain a polar action on  $\mathbb{C}H^n$ , since the whole normalizer  $N_{K_0}(\mathfrak{w})$  acts polarly on  $\mathfrak{w}^\perp$ . Indeed, let  $\mathfrak{s}_\varphi$  be any one-dimensional subspace of  $\mathfrak{w}_\varphi^\perp$  if  $\mathfrak{w}_\varphi^\perp \neq 0$ , and define  $\mathfrak{s} = \bigoplus_{\varphi \in \Phi \cup \{0\}} \mathfrak{s}_\varphi$ . Then  $\mathfrak{s}$  is a section of the action of  $N_{K_0}(\mathfrak{w})$  on  $\mathfrak{w}^\perp$ . The cohomogeneity one examples introduced in [2] correspond to the case where  $\mathfrak{w}^\perp$  has constant Kähler angle,  $\mathfrak{b} = \mathfrak{a}$  and  $Q = N_{K_0}(\mathfrak{w})$ .

*Remark 3.3.* It is straightforward to describe all polar actions of closed subgroups  $Q$  in Theorem 3.1 up to orbit equivalence. In fact, the action of the group  $N_{K_0}(\mathfrak{w})$  is given by the products of the natural representations of the direct factors in (1) on the spaces  $\mathfrak{w}_\varphi^\perp$ . By the main result of Dadok [11], a representation is polar if and only if it is orbit equivalent to the isotropy representation of some Riemannian symmetric space. Therefore, we obtain a representative for each orbit equivalence class of polar actions on  $\mathfrak{w}^\perp$  given by closed subgroups of  $N_{K_0}(\mathfrak{w})$  in the following manner. Given  $\mathfrak{w}$ , for each  $\varphi \in \Phi \cup \{0\}$  choose a Riemannian symmetric space  $M_\varphi$  such that  $\dim M_\varphi = \dim \mathfrak{w}_\varphi^\perp$ . In case  $\pi/2 \in \Phi$ , choose the symmetric spaces such that all of them except possibly  $M_{\pi/2}$  are Hermitian symmetric; in case  $\pi/2 \notin \Phi$ , choose all these symmetric spaces to be Hermitian without exception. Then the isotropy representation of  $\prod_{\varphi \in \Phi \cup \{0\}} M_\varphi$  defines a closed subgroup of  $N_{K_0}(\mathfrak{w})$ , which acts polarly on  $\mathfrak{w}^\perp$  with a section  $\mathfrak{s}$ , which is a totally real subspace of  $\mathfrak{g}_\alpha$ , see [25]. This construction exhausts all orbit equivalent classes of closed subgroups in  $K_0$  leaving  $\mathfrak{w}$  invariant and acting polarly on  $\mathfrak{w}^\perp$  with totally real section.

*Remark 3.4.* There is a curious relation between some of the new examples of polar actions in Theorem 3.1 and certain isoparametric hypersurfaces constructed by the first two authors in [12]. The orbit  $H \cdot o$  of any of the polar actions described in Theorem 3.1 with  $\mathfrak{b} = \mathfrak{a}$  is always a minimal (even austere) submanifold of  $\mathbb{C}H^n$  that satisfies the following property: the distance tubes around it are isoparametric hypersurfaces which are hence foliated by orbits of the  $H$ -action. Moreover, these hypersurfaces have constant principal curvatures if and only if they are homogeneous (i.e. they are the principal orbits of the cohomogeneity one action resulting from choosing  $\mathfrak{q} = \mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$  in Theorem 3.1); this happens precisely when the real subspace  $\mathfrak{w}^\perp$  of  $\mathfrak{g}_\alpha$  has constant Kähler angle. See [12] for more details.

The rest of the paper will be devoted to the proof of the classification result stated in Theorem A. In order to justify the content of the following sections, we will give here a sketch of the proof of Theorem A, and leave the details for the following sections.

Assume that  $H$  is a closed subgroup of  $SU(1, n)$  that acts polarly on  $\mathbb{C}H^n$ . Any subgroup of  $SU(1, n)$  is contained in a maximal proper subgroup  $L$  of  $SU(1, n)$ . We will see that each maximal subgroup of  $SU(1, n)$  either leaves a totally geodesic proper subspace of  $\mathbb{C}H^n$  invariant or it is a parabolic subgroup. In the first case,  $L$  leaves invariant a lower dimensional complex hyperbolic space  $\mathbb{C}H^k$ ,  $k \in \{0, \dots, n-1\}$ , or a real hyperbolic space  $\mathbb{R}H^n$ . The first possibility is tackled in Subsection 4.1, and it follows from this part of the paper that, roughly, the action of  $H$  splits, up to orbit equivalence, as the product of a polar action on the totally geodesic  $\mathbb{C}H^k$ , and a polar action with a fixed point on its normal space. Hence, the problem is reduced to the classification of polar actions on lower dimensional complex hyperbolic spaces, which will allow us to use an induction argument. The second possibility is addressed in Subsection 4.2 where we show that the action of  $H$  is orbit equivalent to the action of  $SO(1, n)$ , which is a cohomogeneity one action whose orbits are tubes around a totally geodesic  $\mathbb{R}H^n$ . If the group  $L$  is parabolic, then its Lie algebra is of the form  $\mathfrak{l} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , for some root space decomposition of  $\mathfrak{su}(1, n)$  (see §2.1). We show in Section 5 that the Lie algebra of  $H$  (up to orbit equivalence) must be of the form  $\mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , with  $\mathfrak{q} \subset \mathfrak{k}_0$ ,  $\mathfrak{b} \subset \mathfrak{a}$ , and  $\mathfrak{w} \subset \mathfrak{g}_\alpha$ , or of the form  $\mathfrak{q} \oplus \mathfrak{a}$ , with  $\mathfrak{q} \subset \mathfrak{k}_0$ . A bit more work leads us to the examples described in Theorem 3.1. Combining the different cases, we will conclude in Section 6 the proof of Theorem A.

#### 4. ACTIONS LEAVING A TOTALLY GEODESIC SUBSPACE INVARIANT

The results in this section show that in order to classify polar actions leaving a totally geodesic complex hyperbolic subspace invariant it suffices to study polar actions on the complex hyperbolic spaces of lower dimensions. We will also show that actions leaving a totally geodesic  $\mathbb{R}H^n$  invariant are orbit equivalent to the cohomogeneity one action of  $SO(1, n)$ . Note that if an isometric action leaves a totally geodesic  $\mathbb{R}H^k$  invariant, it also leaves a totally geodesic  $\mathbb{C}H^k$  invariant.

The following is well-known. Let  $H$  be closed connected subgroup of  $SU(1, n)$ . If the natural action of  $H$  on  $\mathbb{C}H^n$  leaves a totally geodesic proper submanifold of  $\mathbb{C}H^n$  invariant, then there is an element  $g \in SU(1, n)$  such that  $gHg^{-1}$  is contained in one of the subgroups  $S(U(1, k)U(n-k))$  or  $SO(1, n)$  of  $SU(1, n)$ .

**4.1. Actions leaving a totally geodesic complex hyperbolic space invariant.** Let  $L = S(U(1, k)U(n-k)) \subset G = SU(1, n)$ . Let  $M_1$  be the totally geodesic  $\mathbb{C}H^k$  given by the orbit  $L \cdot o$ . Let  $M_2$  be the totally geodesic  $\mathbb{C}H^{n-k}$  which is the image of the normal space  $\nu_o M_1$  under the Riemannian exponential map  $\exp_o$ . Let  $H$  be a closed connected subgroup of  $L$ . Then the  $H$ -action on  $\mathbb{C}H^n$  leaves  $M_1$  invariant and the  $H$ -action on  $\mathbb{C}H^n$  restricted to the isotropy subgroup  $H_o$  leaves  $M_2$  invariant. Let  $\pi_1: L \rightarrow U(1, k)$  and  $\pi_2: L \rightarrow U(n-k)$  be the natural projections.

**Theorem 4.1.** *Assume the  $H$ -action on  $\mathbb{C}H^n$  is nontrivial. Then it is polar if and only if the following hold.*

- (i) *The action of  $H$  on  $M_1$  is polar and nontrivial.*
- (ii) *The action of  $H_o$  on  $M_2$  is polar and nontrivial.*
- (iii) *The action of  $\pi_1(H) \times \pi_2(H_o)$  on  $\mathbb{C}H^n$  is orbit equivalent to the  $H$ -action.*

*Proof.* Assume first that the  $H$ -action on  $\mathbb{C}H^n$  is polar and  $\Sigma$  is a section. Let  $\Sigma_i$  be the connected component of  $\Sigma \cap M_i$  containing  $o$  for  $i = 1, 2$ . Obviously, the  $H$ -orbits on  $M_1$  intersect  $\Sigma_1$  orthogonally. Let  $p$  be an arbitrary point in  $M_1$ . Then the intersection of the orbit  $H \cdot p$  with  $\Sigma$  is non-empty. Let  $q \in (H \cdot p) \cap \Sigma$ . Since  $H$  leaves  $M_1$  invariant, we have that  $q \in M_1$ . Both the Riemannian exponential maps of  $M_1$  and of  $\Sigma$  at the point  $o$  are diffeomorphisms by the Cartan-Hadamard theorem. Hence there is a unique shortest geodesic segment  $\beta$  in  $\Sigma$  connecting  $o$  with  $q$  and there is also a unique shortest geodesic segment  $\gamma$  in  $M_1$  connecting  $o$  with  $q$ . Since

both  $\Sigma$  and  $M_1$  are totally geodesic submanifolds of  $\mathbb{C}H^n$  it follows that  $\beta$  and  $\gamma$  are both also totally geodesic segments of  $\mathbb{C}H^n$  connecting the points  $o$  and  $q$  and must coincide by the Cartan-Hadamard theorem. Hence  $\beta = \gamma$  both lie in  $\Sigma_1$ . This shows that  $\Sigma_1$  meets the  $H$ -orbit through  $p$  (namely, at the point  $q$ ) and completes the proof that (i) holds.

Obviously, the  $H_o$ -orbits on  $M_2$  intersect  $\Sigma_2$  orthogonally. Since  $T_o M_2$  is a submodule of the slice representation of  $H_o$  on  $\nu_o(H \cdot o)$ , the linear  $H_o$ -action on  $T_o M_2$  is polar with section  $T_o \Sigma_2$ . The map  $\exp_o: T_o M_2 \rightarrow M_2$  is an  $H_o$ -equivariant diffeomorphism by the Cartan-Hadamard theorem. In particular, it follows that  $\Sigma_2$  meets all  $H_o$ -orbits in  $M_2$ , since  $T_o \Sigma_2$  meets all  $H_o$ -orbits in  $T_o M_2$ . Thus (ii) holds.

Consider the polar slice representation of  $H_o$  at  $T_o \mathbb{C}H^n$  with section  $T_o \Sigma$ . By [11, Theorem 4], it follows that  $T_o \Sigma = T_o \Sigma_1 \oplus T_o \Sigma_2$ . Since  $H \subset \pi_1(H) \times \pi_2(H)$ , it follows that the actions of the two groups on  $\mathbb{C}H^n$  are orbit equivalent.

Now let us prove the other direction of the equivalence. Assume  $H \subset L$  is a closed subgroup such that (i), (ii) and (iii) hold. Because of (iii) we may replace  $H$  by  $\pi_1(H) \times \pi_2(H)$ . Let  $\Sigma_1$  be the section of the  $H$ -action on  $M_1$  and let  $\Sigma_2$  be the section of  $H$ -action on  $M_2$ . Then by Proposition 2.4, the tangent spaces  $T_o \Sigma_1$  and  $T_o \Sigma_2$  are totally real subspaces of  $T_o \mathbb{C}H^n$ ; moreover,  $\mathbb{C}T_o \Sigma_1 \perp \mathbb{C}T_o \Sigma_2$ . Thus the sum  $T_o \Sigma_1 \oplus T_o \Sigma_2$  is totally real Lie triple system in  $T_o \mathbb{C}H^n$ . Let  $\Sigma$  be the corresponding totally geodesic submanifold.

Using Proposition 2.3, we will show that the  $H$ -action on  $\mathbb{C}H^n$  is polar and  $\Sigma$  is a section. Consider the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  with respect to  $o \in \mathbb{C}H^n$ . We have  $\mathfrak{p} = T_o M_1 \oplus T_o M_2$ . Furthermore, the direct sum decomposition

$$(2) \quad \nu_o(H \cdot o) = (\nu_o(H \cdot o) \cap T_o M_1) \oplus T_o M_2$$

holds. The slice representation of the  $H$ -action on  $M_1$  at the point  $o$  is orbit equivalent to the submodule  $\nu_o(H \cdot o) \cap T_o M_1$  of the slice representation of the  $H$ -action on  $\mathbb{C}H^n$  at  $o$ . The slice representation of the  $H_o$ -action on  $M_2$  at the point  $o$  is orbit equivalent to the submodule  $T_o M_2$  of the slice representation of the  $H$ -action on  $\mathbb{C}H^n$  at  $o$ . By [11, Theorem 4], we conclude that the slice representation of  $H_o$  on  $\nu_o(H \cdot o)$  is polar and a section is  $T_o \Sigma = T_o \Sigma_1 \oplus T_o \Sigma_2$ . We have to show  $\langle [v, w], X \rangle = -B([v, w], \theta(X)) = 0$  for all  $v, w \in T_o \Sigma \subset \mathfrak{p}$  and all  $X \in \mathfrak{h}$ . We may identify the tangent space  $T_o \mathbb{C}H^n = \mathfrak{p}$  with the space of complex  $(n+1) \times (n+1)$ -matrices of the form

$$(3) \quad \left( \begin{array}{c|ccc} 0 & \bar{z}_1 & \dots & \bar{z}_n \\ z_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ z_n & 0 & \dots & 0 \end{array} \right).$$

The subspace  $T_o M_1$  is given by the matrices where  $z_{k+1} = \dots = z_n = 0$ . On the other hand,  $T_o M_2$  consists of those matrices where  $z_1 = \dots = z_k = 0$ . Let  $v, w \in T_o \Sigma_1$ . Then  $[v, w]$  is a matrix all of whose non-zero entries are located in the  $(k+1) \times (k+1)$ -submatrix in the upper left-hand corner, and it follows from (i) and Proposition 2.3 that all vectors in  $\mathfrak{h}$  are orthogonal to  $[v, w]$ . Now assume  $v, w \in T_o \Sigma_2$ . Then  $[v, w]$  is a matrix all of whose non-zero entries are located in the  $(n-k) \times (n-k)$ -submatrix in the bottom right-hand corner. It follows from (ii) and Proposition 2.3 that all vectors in  $\mathfrak{h}$  are orthogonal to  $[v, w]$ . Finally assume  $v \in T_o \Sigma_1$  and  $w \in T_o \Sigma_2$ . In this case, the bracket  $[v, w]$  is contained in the orthogonal complement of the Lie algebra of  $L$  in  $\mathfrak{su}(1, n)$ ; in particular,  $[v, w]$  is orthogonal to  $\mathfrak{h}$ . We conclude that the  $H$ -action on  $\mathbb{C}H^n$  is polar by Proposition 2.3.  $\square$

**4.2. Actions leaving a totally geodesic real hyperbolic space invariant.** Now we assume that the polar action leaves a totally geodesic  $\mathbb{R}H^n$  invariant. We have:

**Theorem 4.2.** *Assume that  $H$  is a closed subgroup of  $SO(1, n) \subset SU(1, n)$ . If the  $H$ -action on  $\mathbb{C}H^n$  is polar and nontrivial, then it is orbit equivalent to the  $SO(1, n)$ -action on  $\mathbb{C}H^n$ ; in particular, it is of cohomogeneity one.*

*Proof.* This proof is divided in three steps.

*Claim 1.* The group  $H$  induces a homogeneous polar foliation on the totally geodesic submanifold  $\mathbb{R}H^n$  given by the  $SO(1, n)$ -orbit through  $o$ .

Let  $M_1$  be the totally geodesic  $\mathbb{R}H^n$  given by the  $SO(1, n)$ -orbit through  $o$ . Obviously, the  $H$ -action leaves  $M_1$  invariant. Assume the  $H$ -action on  $M_1$  has a singular orbit  $H \cdot p$ , where  $p = g(o) \in M_1$ . Consider the action of  $H'$  on  $\mathbb{C}H^n$ , where  $H'$  is the conjugate subgroup  $H' = gHg^{-1}$  of  $SU(1, n)$ . The action of  $H'$  is conjugate to the  $H$ -action on  $\mathbb{C}H^n$ , hence polar. We have the splitting (2) for the normal space of the  $H'$ -orbit through  $o$  as in the proof of Lemma 4.1, where in this case  $M_2$  is the totally geodesic  $\mathbb{R}H^n$  such that  $T_o M_2 = i(T_o M_1)$ . Since  $o$  is a singular orbit of the  $H'$ -action on  $M_1$ , the slice representation of  $H'_o$  on  $V = \nu_o(H' \cdot o) \cap T_o M_1$  is nontrivial. The space  $T_o M_1$  consists of all matrices in (3) where the entries  $z_1, \dots, z_n$  are real. Consequently, the space  $iV$  is contained in the normal space  $\nu_o(H' \cdot o)$  and it follows that the slice representation of  $H'_o$  with respect to the  $H'$ -action on  $\mathbb{C}H^n$  contains the submodule  $V \oplus iV$  with two equivalent nontrivial  $H'_o$ -representations and is hence non-polar by [17, Lemma 2.9], a contradiction. Hence the  $H$ -action on  $M_1$  does not have singular orbits, i.e.  $H$  induces a homogeneous foliation on  $M_1$ .

*Claim 2.* The homogeneous polar foliation induced on the invariant totally geodesic real hyperbolic space consists of only one leaf or all the leaves are points.

Consider the point  $o \in M_1$  as in the proof of Claim 1. The tangent space of  $M_1$  at  $o$  splits as

$$T_o M_1 = T_o(H \cdot o) \oplus (\nu_o(H \cdot o) \cap T_o M_1).$$

The action of the isotropy group  $H_o$  on  $T_o M_1$  respects this splitting. Moreover, the action is trivial on  $V = \nu_o(H \cdot o) \cap T_o M_1$ , as this is a submodule of the slice representation at  $o$ , which lies in a principal orbit of the  $H$ -action on  $M_1$ . It follows that the action of  $H_o$  on  $iV$  is trivial as well and the only possibly nontrivial submodule of the slice representation at  $o$  is  $iW$ , where we define  $W = T_o(H \cdot o)$ . It follows that the action of the isotropy group  $H_o$  on  $iW$  is polar by Proposition 2.3. Let  $\Sigma'$  be a section of this action. Let  $\Sigma$  be a section of the  $H$ -action on  $\mathbb{C}H^n$ . Then we have

$$T_o \Sigma = V \oplus iV \oplus \Sigma'.$$

By Proposition 2.4,  $\Sigma$  is either totally real or  $\Sigma = \mathbb{C}H^n$ . In the first case,  $V$  must be 0, so the action of  $H$  on  $M_1$  is transitive. In the second case, the action of  $H$  on  $\mathbb{C}H^n$  is trivial.

*Claim 3.* The  $H$ -action on  $\mathbb{C}H^n$  is orbit equivalent to the  $SO(1, n)$  action.

Assume the  $H$ -action is nontrivial and polar with section  $\Sigma$ . We will use the notation of Subsection 2.1. By Claim 2,  $H$  acts transitively on  $M_1 = \mathbb{R}H^n$ . By Lemma 2.2, the tangent space  $T_o(H \cdot o) = T_o M_1$  coincides with  $\mathfrak{a} \oplus (1 - \theta)\mathfrak{g}_\alpha^\mathbb{R}$ , where  $\mathfrak{g}_\alpha^\mathbb{R}$  is a totally real subspace of the root space  $\mathfrak{g}_\alpha$  satisfying  $\mathbb{C}\mathfrak{g}_\alpha^\mathbb{R} = \mathfrak{g}_\alpha$ . Moreover  $\nu_o M_1 = i(T_o M_1)$ . The action of the isotropy subgroup  $H_o = H \cap K$  on  $\nu_o M_1$  by the slice representation is polar with section  $T_o \Sigma$ . Since  $iB \in \nu_o M_1$ , by conjugating the section with a suitable element in  $H_o$  we can then assume that  $iB \in T_o \Sigma$ .

According to [7, Proposition 2.2], the group  $H$  contains a solvable subgroup  $S$  which acts transitively on  $M_1 = \mathbb{R}H^n$ . Since  $S$  is solvable, it is contained in a Borel subgroup of  $SO(1, n)$ . As shown in the proof of [6, Proposition 4.2], we may assume that the Lie algebra of such a Borel subgroup is maximally noncompact, i.e. its Lie algebra is  $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^\mathbb{R}$ , where  $\mathfrak{t}$  is an abelian subalgebra of  $\mathfrak{k} \cap \mathfrak{so}(n)$  such that  $\mathfrak{t} \oplus \mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{so}(1, n)$ , see [23]. Note that the Cartan decomposition of  $\mathfrak{so}(1, n)$  with respect to the point  $o \in M_1 = \mathbb{R}H^n$  is  $\mathfrak{so}(1, n) = (\mathfrak{k} \cap \mathfrak{so}(1, n)) \oplus \mathfrak{p}^\mathbb{R}$ , where  $\mathfrak{p}^\mathbb{R} = \mathfrak{a} \oplus (1 - \theta)\mathfrak{g}_\alpha^\mathbb{R} \cong T_o M_1$ , and  $\mathfrak{g}_\alpha^\mathbb{R}$  is the only positive root space of  $\mathfrak{so}(1, n)$  with respect to the maximal abelian subalgebra  $\mathfrak{a}$  of  $\mathfrak{p}^\mathbb{R}$ , for a fixed order in the roots.

Now assume the  $H$ -action on  $\mathbb{C}H^n$  is not of cohomogeneity one. Then  $T_o \Sigma \subset \nu_o M_1$  is a Lie triple system containing  $iB$  and a nonzero vector  $iW$  such that  $iB, iW \in \mathfrak{p}$  are orthogonal. By

Lemma 2.2, there is a vector  $W \in \mathfrak{g}_\alpha^\mathbb{R}$  such that  $w = (1 - \theta)W$ . Then, using Lemma 2.1(a), we have

$$[iB, iw] = \frac{1}{2}[(1 - \theta)Z, (1 - \theta)JW] = \frac{1}{2}(1 + \theta)[\theta JW, Z] = \frac{1}{2}(1 + \theta)W.$$

Since  $T_o(S \cdot o) = T_oM_1$ , it follows that the orthogonal projection of the Lie algebra of  $S$  onto  $\mathfrak{p}$  is  $\mathfrak{p}^\mathbb{R} = \mathfrak{a} \oplus (1 - \theta)\mathfrak{g}_\alpha^\mathbb{R}$ . This implies that  $\mathfrak{a} \oplus \mathfrak{g}_\alpha^\mathbb{R}$  is contained in the Lie algebra of  $S$ , and hence, also in  $\mathfrak{h}$ . But then  $W \in \mathfrak{h}$  and

$$\langle [iB, iw], W \rangle = \frac{1}{2}\langle (1 + \theta)W, W \rangle = \frac{1}{2}\langle W, W \rangle \neq 0,$$

so we have arrived at a contradiction with the criterion for polarity in Proposition 2.3.  $\square$

## 5. THE PARABOLIC CASE

As above, let  $G = SU(1, n)$  be the identity connected component of the isometry group of  $\mathbb{C}H^n$ , and  $K = S(U(1)U(n))$  the isotropy group at some point  $o$ . Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition of the Lie algebra of  $G$  with respect to  $o$ , and choose a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ . As usual we consider  $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , where  $\alpha$  is a simple positive restricted root. The normalizer of  $\mathfrak{n}$  in  $\mathfrak{k}$  is denoted by  $\mathfrak{k}_0$ . Then  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$  is a maximal parabolic subalgebra, and a maximal parabolic subgroup can be written as the semi-direct product  $K_0AN$ .

The aim of this section is to prove the following decomposition theorem.

**Theorem 5.1.** *Let  $H$  be a connected closed subgroup of  $K_0AN$  acting polarly and nontrivially on  $\mathbb{C}H^n$ . Then the action of  $H$  is orbit equivalent to the action of a subgroup of  $K_0AN$  whose Lie algebra can be written as one of the following:*

- (a)  $\mathfrak{q} \oplus \mathfrak{a}$ , where  $\mathfrak{q}$  is a subalgebra of  $\mathfrak{k}_0$ .
- (b)  $\mathfrak{q} \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , where  $\mathfrak{w}$  is a subspace of  $\mathfrak{g}_\alpha$ , and  $\mathfrak{q}$  is a subalgebra of  $\mathfrak{k}_0$ .
- (c)  $\mathfrak{q} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , where  $\mathfrak{w}$  is a subspace of  $\mathfrak{g}_\alpha$ , and  $\mathfrak{q}$  is a subalgebra of  $\mathfrak{k}_0$ .

Let  $Q$  be a maximal compact subgroup of  $H$ . Any two maximal compact subgroups of a connected Lie group  $H$  are connected and conjugate by an element of  $H$  [24, p. 148–149]. By Cartan's fixed point theorem,  $Q$  fixes a point  $p \in \mathbb{C}H^n$ , and hence  $Q = H_p$ , the isotropy group of  $H$  at  $p$ . Since  $AN$  acts simply transitively on  $\mathbb{C}H^n$ , we can take the unique element  $g$  in  $AN$  such that  $g(o) = p$ , and consider the group  $H' = I_{g^{-1}}(H) = g^{-1}Hg$ , whose action on  $\mathbb{C}H^n$  is conjugate to the one of  $H$ . Moreover,  $Q' = I_{g^{-1}}(Q) = g^{-1}Qg$  fixes the point  $o$ . Since  $\mathfrak{a} \oplus \mathfrak{n}$  normalizes  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ , we get that  $AN$  normalizes  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ . In particular,  $\text{Ad}(g^{-1})\mathfrak{h} \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$  and therefore  $H' \subset K_0AN$ . Since we are interested in the study of polar actions up to orbit equivalence, it is not restrictive to assume that the group  $H \subset K_0AN$  acting polarly on  $\mathbb{C}H^n$  admits a maximal connected compact subgroup  $Q$  that fixes the point  $o$ , and hence  $Q \subset K_0$ . We will assume this from now on in this section.

As a matter of notation, given two subspaces  $\mathfrak{m}$ ,  $\mathfrak{l}$ , and a vector  $v$  of  $\mathfrak{g}$ , by  $\mathfrak{m}_\mathfrak{l}$  (resp. by  $v_\mathfrak{l}$ ) we will denote the orthogonal projection of  $\mathfrak{m}$  (resp. of  $v$ ) onto  $\mathfrak{l}$ .

The crucial part of the proof of Theorem 5.1 is contained in the following assertion:

**Proposition 5.2.** *Let  $H$  be a connected closed subgroup of  $K_0AN$  acting polarly on  $\mathbb{C}H^n$ . Let  $Q$  be a maximal subgroup of  $H$  that fixes the point  $o \in \mathbb{C}H^n$ . Let  $\mathfrak{b}$  be a subspace of  $\mathfrak{a}$ ,  $\mathfrak{w}$  a subspace of  $\mathfrak{g}_\alpha$ , and  $\mathfrak{r}$  a subspace of  $\mathfrak{g}_{2\alpha}$ . Assume that  $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$  is a subalgebra of  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ , and let  $\hat{H}$  be the connected subgroup of  $K_0AN$  whose Lie algebra is  $\hat{\mathfrak{h}}$ . If  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ , then the actions of  $H$  and  $\hat{H}$  are orbit equivalent.*

The proof of Proposition 5.2 is carried out in several steps. We start with a few basic remarks.

Since  $\mathfrak{a}$  and  $\mathfrak{g}_{2\alpha}$  are one dimensional,  $\mathfrak{b}$  is either 0 or  $\mathfrak{a}$ , and  $\mathfrak{r}$  is either 0 or  $\mathfrak{g}_{2\alpha}$ . Moreover, if  $\mathfrak{r} = 0$  then  $\mathfrak{w}$  has to be a totally real subspace of the complex vector space  $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ , so that  $\hat{\mathfrak{h}}$



is a Lie subalgebra. Using the properties of the root space decomposition, it is then easy to check that  $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$  is a subalgebra of  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$  if and only if  $[\mathfrak{q}, \mathfrak{w}] \subset \mathfrak{w}$ .

Let  $\Sigma$  be a section of the action of  $H$  on  $\mathbb{C}H^n$  through  $o \in \mathbb{C}H^n$ , and let  $T_o\Sigma$  be its tangent space at  $o$ . The normal space of the orbit through the origin is  $\nu_o(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{p}_\alpha \ominus (1 - \theta)\mathfrak{w}) \oplus (\mathfrak{p}_{2\alpha} \ominus (1 - \theta)\mathfrak{r})$ . Since  $[\mathfrak{k}_0, \mathfrak{a}] = [\mathfrak{k}_0, \mathfrak{g}_{2\alpha}] = 0$ ,  $[\mathfrak{k}_0, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ , and  $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$  (orthogonal direct sum of vector subspaces) by Proposition 2.3, it follows that  $\mathfrak{a} \ominus \mathfrak{b} \subset T_o\Sigma$  and  $\mathfrak{p}_{2\alpha} \ominus (1 - \theta)\mathfrak{r} \subset T_o\Sigma$ . Moreover, since sections are totally real by Proposition 2.4, we can write the tangent space at  $o$  of any section as  $T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)\mathfrak{s} \oplus (\mathfrak{p}_{2\alpha} \ominus (1 - \theta)\mathfrak{r})$ , where  $\mathfrak{s}$  is a totally real subspace of  $\mathfrak{g}_\alpha$ , with  $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ . Furthermore, the fact that  $T_o\Sigma$  is totally real, and  $i\mathfrak{a} = \mathfrak{p}_{2\alpha}$  (where  $i$  is the complex structure on  $\mathfrak{p}$ ), implies that  $\mathfrak{a} \ominus \mathfrak{b} = 0$  or  $\mathfrak{p}_{2\alpha} \ominus (1 - \theta)\mathfrak{r} = 0$ , or equivalently,  $\mathfrak{b} = \mathfrak{a}$  or  $\mathfrak{r} = \mathfrak{g}_{2\alpha}$  (that is,  $\mathfrak{a} \subset \mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}}$  or  $\mathfrak{g}_{2\alpha} \subset \mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}}$ ).

Let  $T + aB + U + xZ$  be an arbitrary element of  $\mathfrak{h}$ , with  $T \in \mathfrak{h}_{\mathfrak{k}_0}$ ,  $U \in \mathfrak{w}$ , and  $a, x \in \mathbb{R}$ . Let  $\xi, \eta$  be arbitrary vectors of  $\mathfrak{s}$ . By Proposition 2.3, and since  $\mathfrak{s}$  is totally real, we have, using Lemma 2.1(b):

$$0 = \langle T + aB + U + xZ, [(1 - \theta)\xi, (1 - \theta)\eta] \rangle = -\langle T, (1 + \theta)[\theta\xi, \eta] \rangle = -2\langle [T, \xi], \eta \rangle,$$

from where it follows that  $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$ .

Moreover, if  $T \in \mathfrak{q}$  and  $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ ,  $U \in \mathfrak{w}$  are such that  $S_U + U \in \mathfrak{h}$ , then  $[T, S_U] + [T, U] = [T, S_U + U] \in \mathfrak{h}$ , so  $[T, U] \in \mathfrak{w}$ . In particular, if  $\xi \in \mathfrak{s}$ , then  $0 = \langle [T, U], \xi \rangle = -\langle [T, \xi], U \rangle$ , which proves  $[\mathfrak{q}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ .

Summarizing what we have obtained about sections we can state:

**Lemma 5.3.** *If  $\Sigma$  is a section of the action of  $H$  on  $\mathbb{C}H^n$  through  $o$ , then*

$$T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)\mathfrak{s} \oplus (\mathfrak{p}_{2\alpha} \ominus (1 - \theta)\mathfrak{r}),$$

where  $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$  is a totally real subspace of  $\mathfrak{g}_\alpha$ , and  $\mathfrak{b} = \mathfrak{a}$  or  $\mathfrak{r} = \mathfrak{g}_{2\alpha}$ . Moreover,  $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$ , and  $[\mathfrak{q}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ .

We will need to calculate the isotropy group at certain points.

**Lemma 5.4.** *Let  $\xi \in \mathfrak{g}_\alpha$  and write  $g = \text{Exp}(\lambda\xi)$ , with  $\lambda \in \mathbb{R}$ . Then, the Lie algebra of the isotropy group  $H_p$  of  $H$  at  $p = g(o)$  is  $\mathfrak{h}_p = \mathfrak{h} \cap \text{Ad}(g)\mathfrak{k} = \mathfrak{q} \cap \ker \text{ad}(\xi)$ .*

*Proof.* First notice that  $\mathfrak{h} \cap \text{Ad}(g)\mathfrak{k}$  is the Lie algebra of  $H_p = H \cap I_g(K)$ . Let  $v$  be the unique element in  $\mathfrak{p} = T_o\mathbb{C}H^n$  such that  $\exp_o(v) = p$ . We show that the isotropy group  $H_p$  coincides with the isotropy group of the slice representation of  $Q$  at  $v$ ,  $Q_v$ . By [29, §2] we know that the normal exponential map  $\exp: \nu(H \cdot o) \rightarrow \mathbb{C}H^n$  is an  $H$ -equivariant diffeomorphism. Let  $h \in H_p$ . Since  $\exp_o(v) = p = h(p) = h \exp_o(v) = \exp_{h(o)}(h_*v)$ , we get that  $h(o) = o$  and  $h_*v = v$ , and hence,  $h \in Q_v$ . The  $H$ -equivariance of  $\exp$  also shows the converse inclusion. Therefore  $H_p = Q_v$ .

We can write  $v = aB + b(1 - \theta)\xi$  for certain  $a, b \in \mathbb{R}$ . In fact,  $\text{Exp}(\lambda\xi)(o)$  belongs to the totally geodesic  $\mathbb{R}H^2$  given by  $\exp_o(\mathfrak{a} \oplus \mathbb{R}(1 - \theta)\xi)$ , and  $b \neq 0$  if  $\lambda\xi \neq 0$ . Then, the Lie algebra of  $H_p = Q_v$  is  $\{T \in \mathfrak{q} : [T, aB + b(1 - \theta)\xi] = 0\} = \{T \in \mathfrak{q} : [T, \xi] = 0\}$ , which is  $\mathfrak{q} \cap \ker \text{ad}(\xi)$ .  $\square$

By definition, we say that a vector  $\xi \in \mathfrak{s}$  is *regular* if  $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ . We have

**Lemma 5.5.** *The set  $\{\xi \in \mathfrak{s} : \xi \text{ is regular}\}$  is an open dense subset of  $\mathfrak{s}$ .*

*Proof.* An element of  $T_o\Sigma$  can be written, according to Lemma 5.3, as  $v = aB + (1 - \theta)\xi + x(1 - \theta)Z$  where  $a, x \in \mathbb{R}$ , and  $\xi \in \mathfrak{s}$ . We have  $[\mathfrak{q}, v] = (1 - \theta)[\mathfrak{q}, \xi]$  and  $\nu_o(H \cdot o) \ominus T_o\Sigma = (1 - \theta)(\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}))$ . An element of  $T_o\Sigma$  is regular (that is, belongs to a principal orbit of the slice representation  $Q \times \nu_o(H \cdot o) \rightarrow \nu_o(H \cdot o)$ ) if and only if  $[\mathfrak{q}, v] = \nu_o(H \cdot o) \ominus T_o\Sigma$ . The previous equalities, and the fact that  $(1 - \theta): \mathfrak{g}_\alpha \rightarrow \mathfrak{p}_\alpha$  is an isomorphism implies that  $v$  is regular if and only if  $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ . Since the set of regular points of a section is open and dense, the result follows.  $\square$

**Lemma 5.6.** *For each regular vector  $\xi \in \mathfrak{s}$  we have  $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ .*

*Proof.* Let  $\xi \in \mathfrak{s}$  be a regular vector, that is,  $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ . In order to prove the lemma, it is enough to show that  $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ , since  $\mathfrak{q} \subset \mathfrak{h}_{\mathfrak{k}_0}$  and, by Lemma 5.3,  $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$ .

First, consider the case  $\mathfrak{r} = 0$ . By Lemma 5.3,  $T_o\Sigma = (1 - \theta)\mathfrak{s} \oplus \mathbb{R}(1 - \theta)Z$  for each section  $\Sigma$  through  $o$ , where  $\mathfrak{s}$  is some totally real subspace of  $\mathfrak{g}_\alpha$ . By Proposition 2.3 we have  $\nu_o(H \cdot o) = \text{Ad}(Q)(T_o\Sigma)$  and, thus, for any  $\eta \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$  we can find a section  $\Sigma$  through  $o$  such that  $\eta \in \mathfrak{s}$  by conjugating by a suitable element in  $Q$ . Then using Lemma 2.1, we have that  $(1 + \theta)J\eta = [(1 - \theta)\eta, (1 - \theta)Z] \in [T_o\Sigma, T_o\Sigma]$ . Let  $W \in \mathfrak{w}$  and  $T_W \in \mathfrak{h}_{\mathfrak{k}_0}$  be such that  $T_W + W \in \mathfrak{h}$ . Since by Proposition 2.3 we have  $\langle \mathfrak{h}, [T_o\Sigma, T_o\Sigma] \rangle = 0$ , then  $0 = \langle T_W + W, (1 + \theta)J\eta \rangle = \langle W, J\eta \rangle$ . We have then shown that  $J(\mathfrak{g}_\alpha \ominus \mathfrak{w})$  is orthogonal to  $\mathfrak{w}$ , that is,  $\mathfrak{g}_\alpha \ominus \mathfrak{w}$  is a complex subspace of  $\mathfrak{g}_\alpha$ . Since  $\mathfrak{w}$  is totally real, we deduce  $\mathfrak{w} = 0$ . But then  $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$  holds trivially.

For the rest of the proof, we assume that  $\mathfrak{r} = \mathfrak{g}_{2\alpha}$ .

Let  $T_B \in \mathfrak{h}_{\mathfrak{k}_0}$  and  $a \in \mathbb{R}$  such that  $T_B + aB \in \mathfrak{h}$ . Note that, if  $\mathfrak{b} = 0$ , then  $a = 0$ ,  $T_B \in \mathfrak{q}$  and there is nothing to prove. For each  $U \in \mathfrak{w}$  take an  $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$  with  $S_U + U \in \mathfrak{h}$ . Then  $[T_B, S_U] + [T_B, U] + \frac{a}{2}U = [T_B + aB, S_U + U] \in \mathfrak{h}$ , so  $[T_B, U] + \frac{a}{2}U \in \mathfrak{w}$ , from where  $[T_B, U] \in \mathfrak{w}$ . Hence,  $\langle [T_B, \xi], U \rangle = -\langle \xi, [T_B, U] \rangle = 0$ , so we get  $[T_B, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ .

Now let  $T_Z \in \mathfrak{h}_{\mathfrak{k}_0}$  and  $x \in Z$  with  $T_Z + xZ \in \mathfrak{h}$ . For each  $U \in \mathfrak{w}$  take an  $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$  with  $S_U + U \in \mathfrak{h}$ . Then  $[T_Z, S_U] + [T_Z, U] = [T_Z + xZ, S_U + U] \in \mathfrak{h}$ , so  $[T_Z, U] \in \mathfrak{w}$ . As above, we conclude  $[T_Z, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ .

Finally, we have to prove that for each  $U \in \mathfrak{w}$ , if  $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$  is such that  $T_U + U \in \mathfrak{h}$ , then  $[T_U, \xi] \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ . This will require some effort.

Let  $U \in \mathfrak{w}$  and  $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$  with  $T_U + U \in \mathfrak{h}$ . By Lemma 5.3,  $[T_U, \xi] \in \mathfrak{g}_\alpha \ominus \mathfrak{s} = \mathfrak{w} \oplus (\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}))$ . Since  $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ , we can find an  $S \in \mathfrak{q}$  so that  $[T_U + S, \xi] \in \mathfrak{w}$ . Therefore we can define the map

$$F_\xi: \mathfrak{w} \rightarrow \mathfrak{w}, \quad U \mapsto [T_U, \xi], \quad \text{where } T_U \in \mathfrak{h}_{\mathfrak{k}_0}, T_U + U \in \mathfrak{h}, \text{ and } [T_U, \xi] \in \mathfrak{w}.$$

The map  $F_\xi$  is well-defined. Indeed, if  $T_U, S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ ,  $U \in \mathfrak{w}$ ,  $T_U + U, S_U + U \in \mathfrak{h}$ , and  $[T_U, \xi], [S_U, \xi] \in \mathfrak{w}$ , then  $T_U - S_U \in \mathfrak{q}$ , so  $[T_U, \xi] - [S_U, \xi] = [T_U - S_U, \xi] \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$ , and  $[T_U, \xi] - [S_U, \xi] \in \mathfrak{w}$ . Hence  $[T_U, \xi] = [S_U, \xi]$ . It is also easy to check that  $F_\xi$  is linear.

Furthermore,  $F_\xi$  is self-adjoint. To see this, let  $T_U, S_V \in \mathfrak{h}_{\mathfrak{k}_0}$ ,  $U, V \in \mathfrak{w}$ , with  $T_U + U, S_V + V \in \mathfrak{h}$ , and  $[T_U, \xi], [S_V, \xi] \in \mathfrak{w}$ . Then we have

$$\begin{aligned} 0 &= \langle [T_U + U, S_V + V], \xi \rangle = \langle [T_U, V], \xi \rangle - \langle [S_V, U], \xi \rangle = -\langle V, [T_U, \xi] \rangle + \langle U, [S_V, \xi] \rangle \\ &= -\langle F_\xi(U), V \rangle + \langle F_\xi(V), U \rangle. \end{aligned}$$

Assume now that  $F_\xi \neq 0$ . Then  $F_\xi$  admits an eigenvector  $U \in \mathfrak{w}$  with nonzero eigenvalue  $\lambda \in \mathbb{R}$ :  $F_\xi(U) = \lambda U \neq 0$ . We will get a contradiction with this.

Let  $g = \text{Exp}(-\frac{1}{\lambda}\xi)$ , and consider  $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $T_U + U \in \mathfrak{h}$  and  $F_\xi(U) = [T_U, \xi] = \lambda U$ . We also consider an element  $S \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $S + Z \in \mathfrak{h}$  and  $[S, \xi] = 0$ ; this is possible because  $[S, \xi] \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}) = [\mathfrak{q}, \xi]$  and  $\mathfrak{q} \subset \mathfrak{h}$ . If we define  $R = T_U - \frac{1}{4\lambda} \langle J\xi, U \rangle S \in \mathfrak{h}_{\mathfrak{k}_0}$ , then we have

$$\begin{aligned} \text{Ad}(g)R &= e^{-\frac{1}{\lambda} \text{ad}(\xi)} R = T_U - \frac{1}{\lambda} [\xi, T_U] + \frac{1}{2\lambda^2} [\xi, [\xi, T_U]] - \frac{1}{4\lambda} \langle J\xi, U \rangle S \\ &= (T_U + U) - \frac{1}{4\lambda} \langle J\xi, U \rangle (S + Z) \in \mathfrak{h} \cap \text{Ad}(g)(\mathfrak{k}). \end{aligned}$$

However,  $\text{Ad}(g)R \notin \mathfrak{q} \cap \ker \text{ad}(\xi)$ . By virtue of Lemma 5.4, this gives a contradiction. Thus we must have  $F_\xi = 0$ , from where the result follows.  $\square$

**Lemma 5.7.** *The subspace  $\mathfrak{h}_{\mathfrak{k}_0}$  is a subalgebra of  $\mathfrak{k}_0$  and  $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{w}] \subset \mathfrak{w}$ .*

*Proof.* If  $T + aB + U + xZ, S + bB + V + yZ \in \mathfrak{h}$ , with  $T, S \in \mathfrak{h}_{\mathfrak{k}_0}$ ,  $U, V \in \mathfrak{w}$ , and  $a, b, x, y \in \mathbb{R}$ , then the bracket  $[T + aB + U + xZ, S + bB + V + yZ] = [T, S] + [T, V] - [S, U] + \frac{a}{2}V - \frac{b}{2}U + (\frac{1}{2} \langle JU, V \rangle + ay - bx)Z$  belongs to  $\mathfrak{h}$ . In particular  $[T, S] \in \mathfrak{h}_{\mathfrak{k}_0}$ , so  $\mathfrak{h}_{\mathfrak{k}_0}$  is a Lie subalgebra of  $\mathfrak{k}_0$ . Taking  $U = 0, a = b = x = y = 0$  we obtain that  $[\mathfrak{q}, \mathfrak{w}] \subset \mathfrak{w}$  and hence  $[\mathfrak{q}, \mathfrak{g}_\alpha \ominus \mathfrak{w}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ .

Now let  $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ . For any section through  $o$  we have  $\text{Ad}(Q)(T_o\Sigma) = \nu_o(H \cdot o) = (\mathfrak{a} \oplus \mathfrak{b}) \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w}) \oplus (1 - \theta)(\mathfrak{g}_{2\alpha} \ominus \mathfrak{r})$ , and  $(\mathfrak{a} \oplus \mathfrak{b}) \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{r}) \subset T_o\Sigma$  by Lemma 5.3. Hence, for  $(1 - \theta)X \in (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w})$  we can find a section  $\Sigma$  such that  $(1 - \theta)X \in T_o\Sigma$  (after conjugation by an element of  $Q$  if necessary). Then, if  $X$  is regular, Lemma 5.6 implies  $[\mathfrak{h}_{\mathfrak{k}_0}, X] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ . Since the set of regular vectors is dense,  $X$  can always be approximated by a sequence of regular vectors, and hence, by continuity we also obtain  $[\mathfrak{h}_{\mathfrak{k}_0}, X] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$  for non-regular vectors. Therefore,  $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{g}_\alpha \ominus \mathfrak{w}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ . Finally, the skew-symmetry of the elements of  $\text{ad}(\mathfrak{k}_0)$  implies  $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{w}] \subset \mathfrak{w}$ .  $\square$

We can now finish the proof of Proposition 5.2.

*Proof of Proposition 5.2.* The fact that  $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$  is a subalgebra of  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ , and Lemma 5.7, imply that  $\hat{\mathfrak{h}} = \mathfrak{h}_{\mathfrak{k}_0} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$  is a Lie subalgebra of  $\mathfrak{g}$  that contains  $\mathfrak{h}$  and  $\hat{\mathfrak{h}}$ . Let  $\tilde{H}$  be the connected subgroup of  $G$  whose Lie algebra is  $\hat{\mathfrak{h}}$ . Since  $T_o(H \cdot o) = T_o(\tilde{H} \cdot o) = T_o(\hat{H} \cdot o) = \mathfrak{b} \oplus (1 - \theta)\mathfrak{w} \oplus (1 - \theta)\mathfrak{r}$  and  $H \subset \tilde{H}$ ,  $\hat{H} \subset \tilde{H}$ , the orbits through  $o$  of the groups  $H$ ,  $\tilde{H}$ , and  $\hat{H}$  coincide. The slice representations at  $o$  of  $H$  and  $\tilde{H}$  have the same principal orbits. Indeed, for a section  $\Sigma$  through  $o$  and  $v = aB + (1 - \theta)\xi + x(1 - \theta)Z \in T_o\Sigma$  with  $\xi \in \mathfrak{s}$  regular, Lemma 5.6 implies  $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}) = [\mathfrak{q}, \xi]$ . Thus, the tangent spaces at  $v$  of the orbits of the slice representations of  $H$  and  $\tilde{H}$  through  $v$  coincide, and since  $H \subset \tilde{H}$ , both orbits coincide. Then, the slice representations at  $o$  of  $H$  and  $\tilde{H}$  are orbit equivalent. Since the codimension of an orbit of  $H$  (resp. of  $\tilde{H}$ ) through  $\exp_o(v)$  coincides with the codimension of the orbit of the slice representation of  $H$  (resp. of  $\tilde{H}$ ) through  $v \in \nu_o(H \cdot o) = \nu_o(\tilde{H} \cdot o)$ , and since the orbits of  $H$  are contained in the orbits of  $\tilde{H}$ , we conclude that the actions of  $H$  and  $\tilde{H}$  on  $\mathbb{C}H^n$  have the same orbits. Similarly, an analogous argument with  $\hat{H}$  instead of  $H$  allows to show that the actions of  $\hat{H}$  and  $\tilde{H}$  on  $\mathbb{C}H^n$  are orbit equivalent, and this completes the proof.  $\square$

We now proceed with the proof of Theorem 5.1.

Let  $H$  be a closed subgroup of the isometry group of  $\mathbb{C}H^n$  acting polarly on  $\mathbb{C}H^n$ , and assume that the Lie algebra of  $H$  is contained in a maximal parabolic subalgebra  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ . As we argued at the beginning of this section, there is a maximal compact subgroup  $Q$  of  $H$ , and we can assume that  $o \in \mathbb{C}H^n$  is a fixed point of  $Q$ , that is, the isotropy group of  $H$  at  $o$  is  $Q$ . We are now interested in  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}}$ , the orthogonal projection of  $\mathfrak{h}$  on  $\mathfrak{a} \oplus \mathfrak{n}$ . It is clear that  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}}$  can be written in one of the following forms:  $\mathfrak{w}$ ,  $\mathbb{R}(B + X) \oplus \mathfrak{w}$ ,  $\mathbb{R}(B + X + xZ) \oplus \mathfrak{w}$  (with  $x \neq 0$ ),  $\mathfrak{w} \oplus \mathbb{R}(Y + Z)$ , or  $\mathbb{R}(B + X) \oplus \mathfrak{w} \oplus \mathbb{R}(Y + Z)$ , where  $\mathfrak{w} \subset \mathfrak{g}_\alpha$ , and  $X, Y \in \mathfrak{g}_\alpha$ .

In order to conclude the proof of Theorem 5.1 we deal with these five possibilities separately.

Case 1:  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{w}$ , with  $\mathfrak{w}$  a subspace of  $\mathfrak{g}_\alpha$ .

Here  $\mathfrak{h}$  is in the hypotheses of Proposition 5.2, and it readily follows from Lemma 5.3 that this case is not possible.

Case 2:  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathbb{R}(B + X) \oplus \mathfrak{w}$ , with  $\mathfrak{w}$  a subspace of  $\mathfrak{g}_\alpha$ , and  $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ .

Assume first that  $X \neq 0$ . Then,  $\nu_o(H \cdot o) = \mathbb{R}(-\|X\|^2 B + (1 - \theta)X) \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w}) \oplus \mathfrak{p}_{2\alpha}$ . Let  $\Sigma$  be a section through  $o$ . Since  $T_o\Sigma \subset \nu_o(H \cdot o)$ ,  $[\mathfrak{q}, -\|X\|^2 B + (1 - \theta)X] \subset \mathfrak{p}_\alpha$ ,  $[\mathfrak{q}, \mathfrak{p}_{2\alpha}] = 0$ , and  $[\mathfrak{q}, \mathfrak{p}_\alpha] \subset \mathfrak{p}_\alpha$ , we get that  $[\mathfrak{q}, T_o\Sigma]$  is orthogonal to  $\mathfrak{a}$  and  $\mathfrak{p}_{2\alpha}$ . As  $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$  (orthogonal direct sum) by Proposition 2.3, we readily get that  $\mathfrak{p}_{2\alpha} \subset T_o\Sigma$ . Moreover, let  $T \in \mathfrak{h}_{\mathfrak{k}_0}$  be such that  $T + B + X \in \mathfrak{h}$ ; then  $T + B + X$  is orthogonal to  $[\mathfrak{q}, T_o\Sigma]$ , and since  $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$  we obtain that  $X$  is orthogonal to  $[\mathfrak{q}, T_o\Sigma]$ . The fact that the direct sum  $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$  is orthogonal implies that  $-\|X\|^2 B + (1 - \theta)X \in T_o\Sigma$ . However, since  $T_o\Sigma$  is totally real we have

$$0 = \langle i(-\|X\|^2 B + (1 - \theta)X), (1 - \theta)Z \rangle = \langle -\frac{1}{2}\|X\|^2(1 - \theta)Z + (1 - \theta)JX, (1 - \theta)Z \rangle = -2\|X\|^2,$$

which is not possible because  $X \neq 0$ .

Therefore we must have  $X = 0$ , and thus  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w}$ . Note that the fact that  $\mathfrak{h}$  is a subalgebra of  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$  implies that  $\mathfrak{w}$  is a totally real subspace of  $\mathfrak{g}_\alpha$ . We are now in the hypotheses of Proposition 5.2 and, as shown in the proof of Lemma 5.6,  $\mathfrak{w} = 0$ . We conclude that the action of  $H$  is orbit equivalent to the action of the group  $\hat{H}$  whose Lie algebra is  $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a}$ . This corresponds to Theorem 5.1(a).

Case 3:  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathbb{R}(B + X + xZ) \oplus \mathfrak{w}$ , with  $\mathfrak{w}$  a subspace of  $\mathfrak{g}_\alpha$ ,  $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ , and  $x \in \mathbb{R}$ ,  $x \neq 0$ .

Let  $g = \exp(xZ) \in G$ , and let  $T + r(B + X + xZ) + V$  be a generic element of  $\mathfrak{h}$ , with  $V \in \mathfrak{w}$ ,  $r \in \mathbb{R}$ . Clearly, since  $g \in AN$  we have  $\text{Ad}(g)(\mathfrak{h}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ . Then, it is easy to obtain

$$\text{Ad}(g)(T + r(B + X + xZ) + V) = T + r(B + X + xZ) + V - rxZ = T + r(B + X) + V.$$

Hence  $(\text{Ad}(g)(\mathfrak{h}))_{\mathfrak{a} \oplus \mathfrak{n}} = \mathbb{R}(B + X) \oplus \mathfrak{w}$ , and  $\text{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$ . Since  $Q$  is a maximal compact subgroup of  $I_g(H) = gHg^{-1}$ , and the orthogonal projection of the Lie algebra of  $I_g(H)$  onto  $\mathfrak{a} \oplus \mathfrak{n}$  is  $\mathbb{R}(B + X) \oplus \mathfrak{w}$ , the new group  $I_g(H)$  satisfies the conditions of Case 2. Therefore, the action of  $H$  is orbit equivalent to the action of the group  $\hat{H}$  whose Lie algebra is  $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a}$ . This also corresponds to Theorem 5.1(a).

Case 4:  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{w} \oplus \mathbb{R}(Y + Z)$ , with  $\mathfrak{w}$  a subspace of  $\mathfrak{g}_\alpha$ , and  $Y \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ .

Assume that  $Y \neq 0$ . Then,  $\nu_o(H \cdot o) = \mathfrak{a} \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w}) \oplus \mathbb{R}(2(1 - \theta)Y - \|Y\|^2(1 - \theta)Z)$ . Let  $\Sigma$  be a section through  $o$ . Then, by Proposition 2.3 we have  $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$  (orthogonal direct sum). Since  $[\mathfrak{q}, 2(1 - \theta)Y - \|Y\|^2(1 - \theta)Z] \subset \mathfrak{p}_\alpha$ ,  $[\mathfrak{q}, \mathfrak{a}] = 0$ , and  $[\mathfrak{q}, \mathfrak{p}_\alpha] \subset \mathfrak{p}_\alpha$ , we get that  $[\mathfrak{q}, T_o\Sigma]$  is orthogonal to  $\mathfrak{a}$  and  $\mathfrak{p}_{2\alpha}$ . Then,  $\mathfrak{a} \subset T_o\Sigma$ . On the other hand, if  $T \in \mathfrak{h}_{\mathfrak{k}_0}$  is such that  $T + Y + Z \in \mathfrak{h}$ , then  $T + Y + Z$  is orthogonal to  $[\mathfrak{q}, T_o\Sigma] \subset \nu_o(H \cdot o)$ , and since  $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$  we also obtain that  $Y$  is orthogonal to  $[\mathfrak{q}, T_o\Sigma]$ . Thus,  $2(1 - \theta)Y - \|Y\|^2(1 - \theta)Z \in T_o\Sigma$ . But, since  $T_o\Sigma$  is totally real, we get

$$0 = \langle B, i(2(1 - \theta)Y - \|Y\|^2(1 - \theta)Z) \rangle = \langle B, 2(1 - \theta)JY + 2\|Y\|^2B \rangle = 2\|Y\|^2,$$

which contradicts  $Y \neq 0$ .

Therefore we have  $Y = 0$ , and thus,  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ . We are now in the hypotheses of Proposition 5.2, and we conclude that the action of  $H$  is orbit equivalent to the action of the connected subgroup  $\hat{H}$  of the isometry group of  $\mathbb{C}H^n$  whose Lie algebra is  $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , with  $\mathfrak{w}$  a subspace of  $\mathfrak{g}_\alpha$ . This corresponds to Theorem 5.1(c).

Case 5:  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathbb{R}(B + X) \oplus \mathfrak{w} \oplus \mathbb{R}(Y + Z)$ , with  $\mathfrak{w} \subset \mathfrak{g}_\alpha$ , and  $X, Y \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ .

This final possibility is more involved.

Our first aim is to show that  $Y = 0$ . So, assume for the moment that  $Y \neq 0$ .

**Lemma 5.8.** *We have  $X = \gamma Y + \frac{2}{\|Y\|^2}JY$ , with  $\gamma \in \mathbb{R}$ .*

*Proof.* Assume that  $X$  and  $Y$  are linearly dependent, that is,  $X = \lambda Y$ , with  $\lambda \in \mathbb{R}$ . Then,  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathbb{R}(B + \lambda Y) \oplus \mathfrak{w} \oplus \mathbb{R}(Y + Z)$ , and there exist  $T, S \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $T + B + \lambda Y, S + Y + Z \in \mathfrak{h}$ . Then,

$$[T, S] + [T, Y] - \lambda[S, Y] + \frac{1}{2}Y + Z = [T + B + \lambda Y, S + Y + Z] \in \mathfrak{h}.$$

Since  $[T, Y] - \lambda[S, Y] \in \mathfrak{g}_\alpha \ominus \mathbb{R}Y$  by the skew-symmetry of the elements of  $\text{ad}(\mathfrak{k}_0)$ , we get  $\frac{1}{2}Y + Z \in \mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}}$ , which is not possible.

Therefore, we can assume that  $X$  and  $Y$  are linearly independent vectors of  $\mathfrak{g}_\alpha$ . In particular,  $X \neq 0$ . Take and fix for the rest of the calculations  $T, S \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $T + B + X, S + Y + Z \in \mathfrak{h}$ .

In this case, the normal space to the orbit through the origin  $o$  can be written as

$$\begin{aligned} \nu_o(H \cdot o) = & \mathbb{R}(-\|X\|^2B + (1 - \theta)X - \frac{1}{2}\langle X, Y \rangle(1 - \theta)Z) \oplus (\mathfrak{p}_\alpha \ominus (1 - \theta)(\mathfrak{w} \oplus \mathbb{R}X \oplus \mathbb{R}Y)) \\ & \oplus \mathbb{R}(-\langle X, Y \rangle B + (1 - \theta)Y - \frac{1}{2}\|Y\|^2(1 - \theta)Z). \end{aligned}$$

Let  $\Sigma$  be a section of the action of  $H$  on  $\mathbb{C}H^n$  through the point  $o \in \mathbb{C}H^n$ . By Proposition 2.3 we have  $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$  (orthogonal direct sum). In particular the vectors  $T + B + X$  and  $S + Y + Z$  are orthogonal to  $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$  (because  $[\mathfrak{k}_0, \mathfrak{a}] = [\mathfrak{k}_0, \mathfrak{g}_{2\alpha}] = 0$ ). This implies that  $X$  and  $Y$  are already orthogonal to  $[\mathfrak{q}, T_o\Sigma]$ , and thus, so are  $-\|X\|^2 B + (1 - \theta)X - \frac{1}{2}\langle X, Y \rangle(1 - \theta)Z$  and  $-\langle X, Y \rangle B + (1 - \theta)Y - \frac{1}{2}\|Y\|^2(1 - \theta)Z$ . Hence, they are in  $T_o\Sigma$  and we can write

$$\begin{aligned} T_o\Sigma &= \mathbb{R}(-\|X\|^2 B + (1 - \theta)X - \frac{1}{2}\langle X, Y \rangle(1 - \theta)Z) \\ &\quad \oplus (1 - \theta)\mathfrak{s} \oplus \mathbb{R}(-\langle X, Y \rangle B + (1 - \theta)Y - \frac{1}{2}\|Y\|^2(1 - \theta)Z), \end{aligned}$$

where  $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$  is totally real, and  $\mathbb{C}X \oplus \mathbb{C}Y$  is orthogonal to  $\mathfrak{s}$  (because sections are totally real). The fact that  $T_o\Sigma$  is totally real also implies

$$\begin{aligned} 0 &= \langle i(-\|X\|^2 B + (1 - \theta)(X - \frac{1}{2}\langle X, Y \rangle Z)), -\langle X, Y \rangle B + (1 - \theta)(Y - \frac{1}{2}\|Y\|^2 Z) \rangle \\ (4) \quad &= \langle (1 - \theta)(-\frac{1}{2}\|X\|^2 Z + JX) + \langle X, Y \rangle B, -\langle X, Y \rangle B + (1 - \theta)(Y - \frac{1}{2}\|Y\|^2 Z) \rangle \\ &= \|X\|^2\|Y\|^2 - \langle X, Y \rangle^2 + 2\langle JX, Y \rangle. \end{aligned}$$

Now, using Lemma 2.1(a), and (4), we compute

$$\begin{aligned} &[-\|X\|^2 B + (1 - \theta)(X - \frac{1}{2}\langle X, Y \rangle Z), -\langle X, Y \rangle B + (1 - \theta)(Y - \frac{1}{2}\|Y\|^2 Z)] \\ &= \frac{1}{2}(1 + \theta)(-2\langle \theta X, Y \rangle + \langle X, Y \rangle X - \|X\|^2 Y - \|Y\|^2 JX + \langle X, Y \rangle JY - \langle JX, Y \rangle Z). \end{aligned}$$

This vector is in  $[T_o\Sigma, T_o\Sigma]$ , which is orthogonal to  $\mathfrak{h}$  by Proposition 2.3, so taking inner product with  $S + Y + Z$ , and using Lemma 2.1(b) and (4), we get  $0 = -2\langle [S, X], Y \rangle - \frac{1}{2}\|Y\|^2\langle JX, Y \rangle$ , which implies

$$(5) \quad \langle [S, X], Y \rangle = -\frac{1}{4}\|Y\|^2\langle JX, Y \rangle.$$

We also have

$$[T + B + X, S + Y + Z] = [T, S] + [T, Y] - [S, X] + \frac{1}{2}Y + (1 + \frac{1}{2}\langle JX, Y \rangle)Z,$$

which is in  $\mathfrak{h}$ , so taking inner product with  $-\langle X, Y \rangle B + (1 - \theta)(Y - \frac{1}{2}\|Y\|^2 Z)$ , and using (5), we obtain

$$0 = -\langle [S, X], Y \rangle + \frac{1}{2}\|Y\|^2 - \|Y\|^2(1 + \frac{1}{2}\langle JX, Y \rangle) = -\frac{1}{2}\|Y\|^2(1 + \frac{1}{2}\langle JX, Y \rangle).$$

Since  $Y \neq 0$ , we get  $\langle JX, Y \rangle = -2$  and thus (4) can be written as

$$\|X\|^2\|Y\|^2 - \langle X, Y \rangle^2 = 4 = \langle JX, Y \rangle^2.$$

Now put  $X = \gamma Y + \delta JY + E$  with  $E$  orthogonal to  $\mathbb{C}Y$ , and  $\gamma, \delta \in \mathbb{R}$ . Then, the previous equation reads  $\|E\|^2\|Y\|^2 = 0$ , which yields  $E = 0$ . This implies the result.  $\square$

Therefore the situation now is  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathbb{R}(B + \gamma Y + \frac{2}{\|Y\|^2} JY) \oplus \mathfrak{w} \oplus \mathbb{R}(Y + Z)$ , with  $\mathbb{C}Y \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ . The normal space can be rewritten as

$$\begin{aligned} \nu_o(H \cdot o) &= \mathbb{R}(-2B + (1 - \theta)JY) \oplus (\mathfrak{p}_\alpha \ominus (1 - \theta)(\mathfrak{w} \oplus \mathbb{C}Y)) \\ &\quad \oplus \mathbb{R}(-\gamma\|Y\|^2 B + (1 - \theta)Y - \frac{1}{2}\|Y\|^2(1 - \theta)Z), \end{aligned}$$

and arguing as above, if  $\Sigma$  is a section through  $o$ , then

$$(6) \quad T_o\Sigma = \mathbb{R}(-2B + (1 - \theta)JY) \oplus (1 - \theta)\mathfrak{s} \oplus \mathbb{R}(-\gamma\|Y\|^2 B + (1 - \theta)Y - \frac{1}{2}\|Y\|^2(1 - \theta)Z),$$

where  $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{C}Y)$  is a totally real subspace of  $\mathfrak{g}_\alpha$ .

**Lemma 5.9.** *If  $S \in \mathfrak{h}_{\mathfrak{k}_0}$  is such that  $S + Y + Z \in \mathfrak{h}$  then  $[S, JY] = \frac{1}{4}\|Y\|^2 Y$ .*

*Proof.* First of all, by the properties of root systems and the skew-symmetry of the elements of  $\text{ad}(\mathfrak{k}_0)$ , we have  $[S, JY] \in \mathfrak{g}_\alpha \ominus \mathbb{R}JY$ .

Lemma 2.1(a) yields

$$(7) \quad \begin{aligned} & [-2B + (1 - \theta)JY, -\gamma\|Y\|^2B + (1 - \theta)(Y - \frac{1}{2}\|Y\|^2Z)] \\ &= (1 + \theta) \left( -[\theta JY, Y] + \left( \frac{1}{2}\|Y\|^2 - 1 \right) Y + \frac{\gamma}{2}\|Y\|^2JY + \frac{1}{2}\|Y\|^2Z \right), \end{aligned}$$

which is a vector in  $[T_o\Sigma, T_o\Sigma]$ .

Take  $U \in \mathfrak{w}$ , and let  $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$  be such that  $T_U + U \in \mathfrak{h}$ . Taking inner product with (7) and using Lemma 2.1(b) we get  $0 = 2\langle [T_U, JY], Y \rangle$ . Using this equality and since  $\mathfrak{h}$  is a Lie subalgebra, we now have

$$0 = \langle [S + Y + Z, T_U + U], -2B + (1 - \theta)JY \rangle = \langle [S, T_U] + [S, U] - [T_U, Y], JY \rangle = \langle [S, U], JY \rangle,$$

and since  $U \in \mathfrak{w}$  is arbitrary,  $[S, JY] \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}JY)$ .

Let  $\xi \in \mathfrak{s}$ . Proposition 2.3 implies

$$0 = \langle S + Y + Z, [-2B + (1 - \theta)JY, (1 - \theta)\xi] \rangle = -\langle S, (1 + \theta)[\theta JY, \xi] \rangle = -2\langle [S, JY], \xi \rangle.$$

Let  $\eta \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{C}Y)$  be an arbitrary vector. Since  $\text{Ad}(Q)(T_o\Sigma) = \nu_o(H \cdot o)$  by Proposition 2.3, we can conjugate the section  $\Sigma$  in such a way that  $\eta \in \mathfrak{s}$ . (Note that  $-2B + (1 - \theta)JY$  and  $-\gamma\|Y\|^2B + (1 - \theta)Y - \frac{1}{2}\|Y\|^2(1 - \theta)Z$  always belong to  $T_o\Sigma$  by (6).) Hence, the equation above shows that  $[S, JY]$  is orthogonal to  $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{C}Y)$ . Altogether this implies  $[S, JY] \in \mathbb{R}Y$ .

Finally, taking inner product of (7) with  $S + Y + Z \in \mathfrak{h}$  we get, using Lemma 2.1(a),  $0 = 2\langle [S, Y], JY \rangle + \frac{1}{2}\|Y\|^4$ , and hence  $[S, JY] = \frac{1}{4}\|Y\|^2Y$  as we wanted.  $\square$

We define  $g = \text{Exp}(-4JY/\|Y\|^2)$ . Recall that the Lie algebra of the isotropy group of  $H$  at  $g(o)$  is  $\mathfrak{h}_{g(o)} = \text{Ad}(g)(\mathfrak{k}) \cap \mathfrak{h} = \mathfrak{q} \cap \ker \text{ad}(JY)$ , according to Lemma 5.4. Let  $S \in \mathfrak{h}_{\mathfrak{k}_0}$  be such that  $S + Y + Z \in \mathfrak{h}$ . Then, Lemma 5.9 yields

$$\text{Ad}(g)(S) = S - \frac{4}{\|Y\|^2}[JY, S] + \frac{8}{\|Y\|^4}[JY, [JY, S]] = S + Y + Z \in \text{Ad}(g)(\mathfrak{k}) \cap \mathfrak{h}.$$

However, it is clear that  $S + Y + Z \notin \mathfrak{q} \cap \ker \text{ad}(JY)$ , which gives a contradiction.

Therefore we have proved that  $Y = 0$ . Thus  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathbb{R}(B + X) \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ . If  $X = 0$  then  $\mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , and we are under the hypotheses of Proposition 5.2, which implies that the action of  $H$  is orbit equivalent to the action of the group  $\hat{H}$  whose Lie algebra is  $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ . This corresponds to Theorem 5.1(b).

For the rest of this case we assume  $X \neq 0$ . Note that the normal space to the orbit through  $o$  is  $\nu_o(H \cdot o) = \mathbb{R}(-\|X\|^2B + (1 - \theta)X) \oplus (\mathfrak{p}_\alpha \ominus (1 - \theta)(\mathfrak{w} \oplus \mathbb{R}X))$ . If  $\Sigma$  is a section through  $o$ , since  $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$  (orthogonal direct sum), and  $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$ , it is easy to deduce, as in previous cases, that

$$T_o\Sigma = \mathbb{R}(-\|X\|^2B + (1 - \theta)X) \oplus (1 - \theta)\mathfrak{s},$$

where  $\mathbb{R}X \oplus \mathfrak{s}$  is a real subspace of  $\mathfrak{g}_\alpha$ .

We define  $g = \text{Exp}(2X)$ . We will show  $(\text{Ad}(g)(\mathfrak{h}))_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$  and  $\text{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$ , which will allow us to apply Proposition 5.2. From now on we take  $T \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $T + B + X \in \mathfrak{h}$ .

Let  $S \in \mathfrak{q}$ . Then  $[S, T] + [S, X] = [S, T + B + X] \in \mathfrak{h}$ , and thus  $[S, X] \in \mathfrak{w}$ . Now let  $U \in \mathfrak{w}$  be an arbitrary vector, and let  $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $S_U + U \in \mathfrak{h}$ . We have  $0 = \langle [S, S_U + U], -\|X\|^2B + (1 - \theta)X \rangle = -\langle [S, X], U \rangle$ , which together with the previous assertion implies  $[S, X] = 0$ . Then  $\text{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$ . In particular this implies that  $Q$  is a maximal compact subgroup of  $I_g(H) = gHg^{-1}$ .

Now we calculate  $[T, X]$ . Let  $U \in \mathfrak{w}$  and  $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $S_U + U \in \mathfrak{h}$ . Then, by the skew-symmetry of the elements of  $\text{ad}(\mathfrak{k}_0)$  we have  $0 = \langle [T + B + X, S_U + U], -\|X\|^2B + (1 - \theta)X \rangle = -\langle [T, X], U \rangle$ , so  $[T, X] \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ . Let now  $\xi \in \mathfrak{s}$ . By Proposition 2.3 we get, using Lemma 2.1(b),  $0 = \langle T + B + X, [-\|X\|^2B + (1 - \theta)X, (1 - \theta)\xi] \rangle = -\langle T, (1 + \theta)[\theta X, \xi] \rangle = -2\langle [T, X], \xi \rangle$ . Using again Proposition 2.3 we have  $\nu_o(H \cdot o) = \text{Ad}(Q)(T_o\Sigma)$ , and thus, for any  $\eta \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$

we can find a section through  $o$  such that  $(1 - \theta)\eta \in T_o\Sigma$  (note that  $-\|X\|^2B + (1 - \theta)X \in T_o\Sigma$  for any section). Hence the previous argument shows  $\langle [T, X], \eta \rangle = 0$ , and altogether this means  $[T, X] = 0$ . Therefore,  $\text{Ad}(g)(T + B + X) = T + B$ , so the projection of this vector onto  $\mathfrak{a} \oplus \mathfrak{n}$  is in  $\mathfrak{a} \subset \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ .

Fix  $U \in \mathfrak{w}$  and  $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $S_U + U \in \mathfrak{h}$ . We calculate  $[S_U, X]$ . For any  $\xi \in \mathfrak{s}$ , by Proposition 2.3 and Lemma 2.1(b), we get  $0 = \langle S_U + U, [-\|X\|^2B + (1 - \theta)X, (1 - \theta)\xi] \rangle = -2\langle [S_U, X], \xi \rangle$ . As in the previous paragraph, one can argue that  $\xi$  can be taken arbitrarily in  $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$  by changing the tangent space to the section, if necessary, by an element of  $\text{Ad}(Q)$ . Hence  $[S_U, X] \in \mathfrak{w}$ , which yields  $\text{Ad}(g)(S_U + U) = S_U + U - 2[S_U, X] + \frac{1}{2}(\langle JX, U \rangle - 2\langle JX, [S_U, X] \rangle)Z$ , and thus, its projection onto  $\mathfrak{a} \oplus \mathfrak{n}$  belongs to  $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ .

Finally, let  $S_Z \in \mathfrak{h}_{\mathfrak{k}_0}$  such that  $S_Z + Z \in \mathfrak{h}$ . For each  $\xi \in \mathfrak{s}$  we obtain  $0 = \langle S_Z + Z, [-\|X\|^2B + (1 - \theta)X, (1 - \theta)\xi] \rangle = -2\langle [S_Z, X], \xi \rangle$ , and since  $\xi$  can be taken to be in  $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$  by a suitable conjugation of the section by an element in  $\text{Ad}(Q)$ , we deduce  $[S_Z, X] \in \mathfrak{w}$ . Hence,  $\text{Ad}(g)(S_Z + Z) = S_Z - 2[S_Z, X] + (1 - \langle JX, [S_Z, X] \rangle)Z$ , and the orthogonal projection of this vector onto  $\mathfrak{a} \oplus \mathfrak{n}$  belongs to  $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ .

These last calculations show that  $(\text{Ad}(g)(\mathfrak{h}))_{\mathfrak{a} \oplus \mathfrak{n}} \subset \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ . Since  $g \in AN$  normalizes  $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ , we have that  $\text{Ad}(g)(\mathfrak{h}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ . Then the kernel of the projection of  $\text{Ad}(g)(\mathfrak{h})$  onto  $\mathfrak{a} \oplus \mathfrak{n}$  is precisely  $\text{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0$ , which is a compact subalgebra of  $\text{Ad}(g)(\mathfrak{h})$  containing  $\mathfrak{q} = \text{Ad}(g)(\mathfrak{q})$ . By the maximality of  $\mathfrak{q}$  we get that  $\text{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0 = \mathfrak{q}$ . But then by elementary linear algebra

$$\begin{aligned} \dim(\text{Ad}(g)(\mathfrak{h}))_{\mathfrak{a} \oplus \mathfrak{n}} &= \dim \text{Ad}(g)(\mathfrak{h}) - \dim(\text{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0) \\ &= \dim \mathfrak{h} - \dim \mathfrak{q} = \dim \mathfrak{h}_{\mathfrak{a} \oplus \mathfrak{n}} = \dim(\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}). \end{aligned}$$

All in all we have shown that the Lie algebra  $\text{Ad}(g)(\mathfrak{h})$  of  $I_g(H) = gHg^{-1}$  satisfies  $(\text{Ad}(g)(\mathfrak{h}))_{\mathfrak{a} \oplus \mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , and that  $Q$  is a maximal compact subgroup of  $I_g(H)$ . Therefore, we can apply Proposition 5.2 to  $I_g(H)$ . This implies that the action of  $H$  on  $\mathbb{C}H^n$  is orbit equivalent to the action of the group  $\hat{H}$  whose Lie algebra is  $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ . This corresponds to Theorem 5.1(b).

Altogether, we have concluded the proof of Theorem 5.1.

## 6. PROOF OF THE MAIN RESULT

In this section we conclude the proof of Theorem A using the results of Sections 4 and 5.

*Proof of Theorem A.* The actions described in part (i) are polar by virtue of Lemma 4.1 and Theorem 4.2, whereas the polarity of the actions in part (ii) follows from Theorem 3.1.

Since  $\mathfrak{a} \subset \mathfrak{p}$ , the actions in (ii) with  $\mathfrak{b} = \mathfrak{a}$  contain the geodesic line  $\exp_o \mathfrak{a}$ . On the other hand, a horospherical foliation of  $\mathbb{C}H^n$  is given by the action on  $\mathbb{C}H^n$  of the connected subgroup  $N$  of  $G$  with Lie algebra  $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ , see [7]. This shows that an orbit of minimal type for the actions with  $\mathfrak{b} = 0$  is contained in a horosphere.

An action of a subgroup  $H$  of the isometry group  $I(M)$  of a Riemannian manifold  $M$  is proper if and only if  $H$  is a closed subgroup of  $I(M)$ . Hence we may assume  $H \subset SU(1, n)$  is closed. Since the polarity of the action depends only on the Lie algebra of  $H$  by Proposition 2.3, we may assume that  $H$  is connected.

Thus, let  $H$  be a connected closed subgroup of  $SU(1, n)$  acting polarly on  $\mathbb{C}H^n$ . The Lie algebra  $\mathfrak{h}$  of  $H$  is contained in a maximal subalgebra of  $\mathfrak{su}(1, n)$ . By [24, Theorem 1.9, Ch. 6], the maximal nonsemisimple subalgebras of a semisimple real Lie algebra are parabolic or coincide with the centralizer of a pseudotoric subalgebra. (A subalgebra  $\mathfrak{t} \subset \mathfrak{g}$  is called *pseudotoric* if  $\text{Exp ad } \mathfrak{t} \subset \text{Int } \mathfrak{g}$  is a torus.) The maximal subalgebras of simple real Lie algebras which are centralizers of pseudotoric subalgebras have been classified in [26]. However, it is easy to determine them in the case of  $\mathfrak{su}(1, n)$ . Indeed, it follows from [24, Theorem 3.3, Ch. 4] that for all pseudotoric subalgebras  $\mathfrak{t}$  of  $\mathfrak{su}(1, n)$  there is an element  $g \in SU(1, n)$  such that  $\text{Ad}(g)\mathfrak{t}$  is contained in the subalgebra comprised of all diagonal matrices in  $\mathfrak{su}(1, n)$ . Since we are interested in maximal

subalgebras which are centralizers of pseudotoric subalgebras  $\mathfrak{t}$  we may restrict ourselves to one-dimensional pseudotoric subalgebras  $\mathfrak{t}$ . For such a subalgebra we have  $\mathfrak{t} = \mathbb{R} \operatorname{diag}(it_0, \dots, it_n)$ , for  $t_0, \dots, t_n \in \mathbb{R}$  such that  $t_0 + \dots + t_n = 0$ . The centralizers of such  $\mathfrak{t}$  are the subalgebras of the form  $\mathfrak{s}(\mathfrak{u}(1, n_1) \oplus \mathfrak{u}(n_2) \oplus \dots \oplus \mathfrak{u}(n_\ell))$  where  $n_1 + \dots + n_\ell = n$ . In particular, any maximal connected subgroup of  $SU(1, n)$  whose Lie algebra is the centralizer of a pseudotoric subalgebra is conjugate to one of the maximal subgroups  $S(U(1, k)U(n - k))$ ,  $k = 0, \dots, n - 1$ .

First, let us assume  $H$  is contained (after conjugation) in a maximal subgroup of the form  $S(U(1, k)U(n - k))$  or in a semisimple maximal subgroup of  $SU(1, n)$ . In both cases, the action of  $H$  on  $\mathbb{C}H^n$  leaves a totally geodesic submanifold invariant; this follows from the Karpelevich-Mostow Theorem [16], [22] (it is obvious in the first case). This situation has been studied in Section 4.

If the action of  $H$  leaves a totally geodesic  $\mathbb{R}H^n$  invariant, then Theorem 4.2 applies and the  $H$ -action is orbit equivalent to the cohomogeneity one action of  $SO(1, n)$ . This corresponds to case (i) with  $k = n$  in Theorem A. If the action of  $H$  leaves a totally geodesic  $\mathbb{R}H^k$  invariant, with  $k < n$ , then it also leaves a totally geodesic  $\mathbb{C}H^k$  invariant.

Let then  $k$  be the smallest complex dimension of a totally geodesic complex hyperbolic subspace left invariant by the  $H$ -action. If  $k = 0$ , then the  $H$ -action has a fixed point. In this case, it follows from [13] that  $H$  is a subgroup of  $S(U(1)U(n)) \cong U(n)$  that corresponds to a polar action on  $\mathbb{C}P^{n-1}$ , and therefore is induced by the isotropy representation of a Hermitian symmetric space. This corresponds to case (i) with  $k = 0$  in Theorem A.

Let us assume from now on that  $k \geq 1$ . Lemma 4.1 guarantees that the  $H$ -action is orbit equivalent to the product action of a closed subgroup  $H_1$  of  $SU(1, k)$  acting polarly on  $\mathbb{C}H^k$  times a closed subgroup  $H_2$  of  $U(n - k)$  acting polarly (and with a fixed point) on  $\mathbb{C}H^{n-k}$ . By assumption, the  $H_1$ -action on  $\mathbb{C}H^k$  does not leave any totally geodesic  $\mathbb{C}H^l$  or  $\mathbb{R}H^l$  with  $l < k$  invariant. Hence, either the  $H_1$ -action on  $\mathbb{C}H^k$  is orbit equivalent to the  $SO(1, k)$ -action on  $\mathbb{C}H^k$ , or  $H_1$  is contained in a maximal parabolic subgroup of  $SU(1, k)$ . The first case corresponds to part (i) with  $k \in \{1, \dots, n\}$ . Note that for  $Q = H_2$ , the  $Q$ -action on  $\mathbb{C}H^{n-k}$  is determined by its slice representation at the fixed point, so  $Q$  acts polarly with a totally real section on  $T_o\mathbb{C}H^{n-k} \cong \mathbb{C}^{n-k}$ .

Let us consider the second case, that is,  $H_1$  is contained in a maximal parabolic subgroup of  $SU(1, k)$ ,  $k \in \{1, \dots, n\}$ . As explained at the beginning of Section 5, we may assume  $\mathfrak{h}_1 \subset \mathfrak{k}_0^1 \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^1 \oplus \mathfrak{g}_{2\alpha}$ , where now  $\mathfrak{g}_\alpha^1$  is a complex subspace of  $\mathfrak{g}_\alpha$  with complex dimension  $k - 1$ , and  $\mathfrak{k}_0^1 \cong \mathfrak{u}(k - 1)$  is the normalizer of  $\mathfrak{a}$  in  $\mathfrak{k} \cap \mathfrak{su}(1, k)$ . It follows that the  $H_1$ -action is orbit equivalent to the action of a closed subgroup of  $SU(1, k)$  with one of the Lie algebras described in Theorem 5.1: (a)  $\mathfrak{q}^1 \oplus \mathfrak{a}$ , (b)  $\mathfrak{q}^1 \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , or (c)  $\mathfrak{q}^1 \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ , where  $\mathfrak{w}$  is a real subspace of  $\mathfrak{g}_\alpha^1$ , and  $\mathfrak{q}^1 \subset \mathfrak{k}_0^1$  normalizes  $\mathfrak{w}$ . Since  $H_2 \subset U(n - k)$  acts on  $\mathbb{C}H^{n-k}$ , we can define  $\mathfrak{q} = \mathfrak{q}^1 \oplus \mathfrak{h}_2$ , which is a subalgebra of  $\mathfrak{k}_0$ . Part (a) of Theorem 5.1 is then a particular case of Theorem A(i) for  $k = 1$ , while parts (b) and (c) of Theorem 5.1 correspond to Theorem A(ii), where  $\mathfrak{b} = \mathfrak{a}$  and  $\mathfrak{b} = 0$ , respectively. Lemma 2.2, Proposition 2.4 and the fact that the slice representation of a polar action is also polar, guarantee that the action of  $\mathfrak{q}$  on the orthogonal complement of  $\mathfrak{w}$  in  $\mathfrak{g}_\alpha$  is polar with a totally real section.  $\square$

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