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Sylow Numbers from Character Tables and Integral Group Rings

W. Kimmerle and I. Köster

1. Introduction

Groups naturally occur as symmetries of objects. The systematic study of finite groups was started by Ludwig Sylow in 1872. The Sylow theorems give the existence and the number of maximal p-subgroups. The composition of the so-called Sylow p-numbers was analyzed by Marshall Hall in [5]. At the end of the 20th century the question arised what influence Sylow p-numbers have on properties of the given group.

In a series of articles finite groups with given Sylow numbers were studied. Florian Luca and Wenbin Guo gave evidence that for certain Sylow numbers the given group has to be soluble [17], [4], see also [18], [15]. Jiping Zhang and Naoki Chigira found an equivalent condition for p-nilpotency [22], [3]. In 2006, Xianhua Li showed in [16] the uniqueness of Sylow numbers for finite simple groups (excluding $B_n(q)$ and $C_n(q)$).

Alexander Moretó gave in [18] a criteria for the existence of nilpotent Hall π-subgroups as a function of Sylow numbers. As nilpotent Hall π-subgroups are determined by the character table (see [12]), this leads to the question, whether Sylow numbers are determined by character tables.

In this note only finite groups G are considered. By $\pi(G)$ we denote the set of all prime divisors p of $|G|$. The Sylow p-number $n_p(G)$ is the number of all Sylow p-subgroups and defined as $n_p(G) = |G : \mathrm{N}_G(P)|$, whereas $\mathrm{N}_G(P)$ is the normalizer of a Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$ in G. The set $\{ n_p(G) \mid p \in \pi(G) \}$ is called Sylow numbers $\mathrm{sn}(G)$.

In the first section we analyze carefully the situation under extensions. Using nilpotent and central extensions this enables us to prove that Sylow numbers of supersoluble groups are determined by their character table. This is an immediate corollary of Theorem 3.4. In the second section we consider groups with isomorphic integral group rings and prove that the Sylow numbers of a p - constrained group are determined by its integral group ring, cf. Theorem 4.2. In particular this shows that the integral group ring of a finite soluble group G determines its Sylow numbers. The reason that we get a stronger result for integral group rings is the so - called F^* - theorem [19], see also [8]. This shows that a finite group G is determined by $\mathbb{Z}G$ up to isomorphism provided the generalized Fitting subgroup $F^*(G)$ is a p - group. This is certainly not the case for character tables. Note that in general $\mathbb{Z}G$ does not determine G up to isomorphism, the smalles counterexample known is a group of order $97^{28} \cdot 2^{21}$ [7]. We remark that we do not know an example of a finite group G such that $X(G)$ does not determine $\mathrm{sn}(G)$.

2. Sylow numbers

As mentioned above G denotes always a finite group. The following result of Marshall Hall yields a formula for Sylow p-numbers by Sylow numbers of certain subgroups and factor groups:

Theorem 2.1. *[5, Theorem 2.1] Let G have a normal subgroup M and assume $P \in \mathrm{Syl}_p(G)$. Then*

$$n_p(G) = n_p(M)n_p(G/M)n_p(\mathrm{N}_{PM}(P \cap M)/(P \cap M)).$$

The theorem contains the following basic observations for group extensions. We note that these observations may be easily established with direct arguments.

Proposition 2.2. *Let G have a normal q-subgroup Q.*

(i) *If $p \neq q$, then $n_p(G) = n_p(G/Q)n_p(PQ)$.*
(ii) *If $p = q$, then $n_p(G) = n_p(G/Q)$.*
(iii) *Let E be a finite central extension of G with kernel K. Then $n_p(E) = n_p(G)$ for each prime p.*

Proof: (i) and (ii) are immediate from 2.1. For (iii) we use induction. So we may assume that K is of prime order.

Let p be a prime and denote by P a Sylow p - subgroup of G. If $P \cap K = \{1\}$ then by (i) follows $n_p(G) = n_p(G/K)n_p(PK)$. But $n_p(PK) = 1$ because K is central. If $P \cap K \neq \{1\}$ then $P \cap K \trianglelefteq G$ and (ii) yields $n_p(G) = n_p(G/(P \cap K))$. □

The structure of Sylow p-numbers only depends on composition factors of G, as stated in the next proposition:

Theorem 2.3. *[5, Theorem 2.2] The number of Sylow p-subgroups is the product of factors of the following two kinds:*

(i) *the number $n_p(X)$ of Sylow p-subgroups of a simple group S and*
(ii) *a prime power q^t dividing the order of a chief factor T of G and $q^t \equiv 1 \mod p$.*

A Sylow number of a soluble group therefore is the product of prime powers q^t which divides the order of a chief factor and each factor is congruent $1 \mod p$.

In order to use induction (especially for soluble groups) with quotient modulo normal subgroups K where $p \notin \pi(K)$ the main problem is to determine the Sylow number of PK. A first basic observation is the following.

Proposition 2.4. *[2, Proposition 1C] Let $K \trianglelefteq G$ and $K \cap P = \{1\}$. Then*

$$n_p(PK) = |K : \mathrm{C}_G(P) \cap K|$$

and

$$n_p(G) = n_p(G/K) \cdot |K : \mathrm{C}_G(P) \cap K|.$$

For the convenience of the reader we include a slightly more direct proof.

Proof: Let $P \in \mathrm{Syl}_p(G)$ and assume that $P^g \in \mathrm{Syl}_p(PK)$. Due to the trivial intersection of P and K and that K has order coprime to P the Schur - Zassenhaus theorem yields $k \in K$ with $P^g = P^k$.

Suppose for $k_1, k_2 \in K$, that $P^{k_1} = P^{k_2}$. Then $k_1 k_2^{-1} \in \mathrm{N}_G(P) \cap K$, i.e. $n_p(PK) = |K : \mathrm{N}_G(P) \cap K|$.

It remains to prove, that $\mathrm{N}_G(P) \cap K = \mathrm{C}_G(P) \cap K$. Let $k \in \mathrm{N}_G(P) \cap K$. It follows that $p^{-1}k^{-1}pk \in K \cap P = \{1\}$ for every $p \in P$. Consequently $pk = kp$ for every $k \in \mathrm{N}_G(P) \cap K$ and every $p \in P$. Thus, the result holds. □

In some special cases it is possible to obtain the Sylow numbers explicitely.

Corollary 2.5. *Let G be a finite Frobenius group with Frobenius kernel K and Frobenius complement H. Let $\mathrm{sn}(H) = \{a_1, \ldots, a_n\}$. Then*

$$\mathrm{sn}(G) = \{1, |K| \cdot a_1, \ldots, |K| \cdot a_n\}.$$

In particular if H is nilpotent then $\mathrm{sn}(G) = \{1, |K|\}$.

Proof: If $p \in \pi(K)$ then $n_p(G) = 1$ because by J. Thompson the Frobenius kernel is nilpotent. If $p \notin \pi(K)$ then we get by Propostion 2.4 that

$$n_p(G) = n_p(G/K) \cdot |K : \mathrm{C}_G(P) \cap K|.$$

Because H acts fixpointfreely on K we have always $\mathrm{C}_G(P) \cap K = 1$. This completes the proof. $\square$

Another obvious consequence of Proposition 2.4 is the following.

Corollary 2.6. *Assume $K \trianglelefteq G$, where $p \notin \pi(K)$. Then the following assertions are equivalent:*
(i) $n_p(G) = n_p(G/K)$,
(ii) $K \subseteq \mathrm{N}_G(P)$,
(iii) $K \subseteq \mathrm{C}_G(P)$.

Especially for induction with respect to soluble groups the following result is helpful.

Proposition 2.7. *Let G be a finite group with non-trivial normal subgroups M and N of coprime order. Suppose that all Sylow numbers of $n_p(G/M)$, $n_p(N)$ and $n_p(G/MN)$ are known. Then $n_p(G)$ is known for each $p \notin \pi(M) \cup \pi(N)$.*

Proof. Because $p \notin \pi(M)$ we get by Proposition 2.6

$$n_p(G) = n_p(G/M) \cdot |M : M \cap \mathrm{C}_G(P)|,$$

where $P \in \mathrm{Syl}_p(G)$ and similarly, because $p \notin \pi(N)$

$$n_p(G/M) = n_p(G/MN) \cdot n_p(N) \cdot |MN/N : \mathrm{C}_{G/N}(PN/N) \cap MN/N|.$$

Consider the restriction $\kappa|_{\mathrm{C}_G(P)\cap M} : \mathrm{C}_G(P) \cap M \to \mathrm{C}_{G/N}(PN/N) \cap MN/N$ of the reduction map $\kappa : G \to G/N$. This map is injective. Suppose for $x \in M$ that $xN \in \mathrm{C}_{G/N}(PN/N)$, i.e. for every $y \in P$ we obtain $y \cdot x = x \cdot y \cdot n$ for some $n \in N$. But as $n = [x, y] \in M$ and $M \cap N = 1$ the map is also surjective and therefore

$$|M : M \cap \mathrm{C}_G(P)| = |MN/N : \mathrm{C}_{G/N}(PN/N) \cap MN/N|.$$

As $n_p(G/M)$, $n_p(N)$ and $n_p(G/MN)$ are known, the result follows. $\square$

3. Character Tables

We first collect some of the known results on ordinary character tables concerning properties of the group G reflected by $\mathrm{X}(G)$.

3.1. *Let G be a finite group.*
(i) *The second orthogonality relations show that $\mathrm{X}(G)$ determines the length of the conjugacy classes of G.*
(ii) *The lattice of normal subgroups may be constructed out of $\mathrm{X}(G)$, see [11, p.23]. The normal subgroups of G are given by intersections of the kernel of the irreducible characters. For each normal subgroup N of G the conjugacy classes in N are determined. Moreover the order of $|N|$ is determined.*
(iii) *Let C be a conjugacy class and $g \in C$. Then by a result of G. Higman [11, Theorem (8.21)] the prime divisors of the order of g may be calculated from $\mathrm{X}(G)$.*
(iv) *For a given $N \trianglelefteq G$ the ordinary character table $\mathrm{X}(G/N)$ may be computed out of the $\mathrm{X}(G)$ by deleting appropriate lines and columns [11, p.24].*

Of course the question whether the character table determines the Sylow numbers has an affirmative answer when the character table determines the group up to isomorphism. This is for example the case when G is semisimple, i.e. the direct product of non-abelian simple groups [13, Satz 6.3], see also [12, Theorems 4 and 5].

Proposition 3.2. *Suppose that the finite group G is quasinilpotent. Then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof. Quasinilpotent groups are central extensions of semisimple groups. By Proposition 2.2 Sylow numbers remain unchanged under central extensions and by 3.1 (iv) and [13] the result follows immediately. $\square$

Lemma 3.3. *Let $K \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$, then it is possible to decide with $\mathrm{X}(G)$ whether the intersection $K \cap \mathrm{C}_G(P)$ is trivial or not.*

Especially, if K is cyclic of prime order q, then $n_p(G)$ may be calculated from $\mathrm{X}(G)$ provided $n_p(G/K)$ is known.

Proof: Regarding the character table we obtain using 3.1 (iii) the conjugacy classes which are contained in K. By 3.1 (i) we can compute $|\mathrm{C}_G(h)|$ for each $h \in K$. If $|P|$ divides $|\mathrm{C}_G(h)|$, then $h \in \mathrm{C}_G(P^g)$ for some $g \in G$. Consequently $h^{g^{-1}}$ centralizes P. Conversely, if a non-trivial element h of K centralizes P, then $|P|$ divides the order of its centralizer.
If $q = p$ then $n_p(G) = n_p(G/K)$. In the other case we use Proposition 2.4. As K has prime order the index $|K : K \cap \mathrm{C}_G(P)|$ equals q if and only if there is a non-trivial conjugacy class $k^G \in K$ such that $|P|$ divides $|\mathrm{C}_G(k)|$. □

Note that the previous result holds for arbitrary cyclic subgroups K. In general we cannot decide by means of the character table whether the whole conjugacy class g^G of g is contained in $\mathrm{C}_G(P)$. For nilpotent normal subgroups of a group we are able to compute the Sylow number of each factor seperately.

Theorem 3.4. *Suppose that the finite group G has a nilpotent normal subgroup N such that G/N is nilpotent then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof: By 3.1 we may assume that the Sylow numbers of all proper quotients of G are given. If $p \in \pi(N)$, then the intersection $P \cap N$, whereas $P \in \mathrm{Syl}_p(G)$, is normal in G, as $P \cap N$ is a characteristic subgroup of N and N is normal in G. By Proposition 2.2 the Sylow p-number remains unchanged if we consider $n_p(G/(P \cap N))$ instead of $n_p(G)$.
So assume that $p \notin \pi(N)$. Then $P \cap N = 1$ and we obtain with Proposition 2.4

$$n_p(G) = n_p(PN).$$

A nilpotent group N is the direct product of its Sylow subgroups, i.e. $N = P_1 \times \ldots \times P_k$, $P_i \in \mathrm{Syl}_{p_i}(N)$ and $p_i \nmid p$ for all i. Consider $PN/\mathrm{C}(P_1)$, where $\mathrm{C}(P_1) \trianglelefteq PN$ denotes the center of P_1. By 2.1 we get

$$n_p(PN) = n_p(PN/\mathrm{C}(P_1)) \cdot n_p(P\mathrm{C}(P_1)).$$

Proposition 2.4 yields $n_p(P\mathrm{C}(P_1)) = |\mathrm{C}(P_1) : \mathrm{C}(P_1) \cap \mathrm{C}_G(P)|$. Assume $a \in \mathrm{C}(P_1) \cap \mathrm{C}_G(P)$. Consequently, as N is nilpotent, we obtain that $a \in \mathrm{C}(N)$. By assumption G/N is nilpotent, therefore for each $\tilde{P} \in \mathrm{Syl}_p(G)$ there exists $n \in N$ such that $\tilde{P} = P^n$. Then it follows $a \in \mathrm{C}_G(\tilde{P})$, as

$$\tilde{P}^a = (P^n)^a \overset{a\in\mathrm{C}(N)}{=} (P^a)^n \overset{a\in\mathrm{C}_G(P)}{=} P^n = \tilde{P}.$$

Let $a^g \in a^G$. Since $P^{a^g} = P$ we see that a^G is contained in $\mathrm{C}_G(P)$. Therefore $|n_p(P\mathrm{C}(P_1))|$ may be computed by the character table. Note that the Sylow number $n_p(PN/\mathrm{C}(P_1)) = n_P(G/C(P_1))$ is given by induction and thus $n_p(G)$ is determined. □

Proposition 3.5. *Suppose that G has a supersoluble normal subgroup N such that each chief factor of N is also a chief factor of G and $n_p(G/N)$ is known for $p \in \pi(G/N)$. Then $n_p(G)$ may be calculated from $\mathrm{X}(G)$.*

Proof. By assumption a maximal normal subgroup M within N is of prime index q in N. Now Lemma 3.3 implies that $n_p(G/M)$ is known. By induction the result follows. □

Corollary 3.6. *Suppose that G is supersoluble. Then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof. The corollary follows immediately from Proposition 3.5 putting $N = G$. □

Next we consider the case of cyclic Sylow subgroups. The following is obvious.

Lemma 3.7. *Suppose that G has a cyclic Sylow p - subgroup and assume that the order of the representatives of the conjugacy classes of p - elements of G in the character table $\mathrm{X}(G)$ is given. Then $n_p(G)$ may be computed from $\mathrm{X}(G)$.*

Proof. Let p^a be the maximal power of p dividing $|G|$. Let g be a p - element of order p^a. The length L of the conjugacy class of g may be calculated from $\mathrm{X}(G)$. Moreover by assumption the number m of conjugacy classes of p - elements of order p^a is given.Let $P = \langle g \rangle$. Then $L = |G/\mathrm{C}_G(P)|$ and because in each conjugacy class of elements of order p^a are precisely $|\mathrm{N}_G(P)/|\mathrm{C}_G(P)|$ elements of P we get that

$$p^a - p^{a-1} = m \cdot \frac{|\mathrm{N}_G(P)|}{|\mathrm{C}_G(P)|},$$

therefore

$$n_p(G) = L \cdot m \cdot \frac{1}{p^{a-1} \cdot (p-1)}.$$

□

Remarks. The character table with the power map on the conjugacy classes (i.e with given headline) is usually called the spectral table $\mathrm{Spec}(G)$ of G. If the spectral table is given then the order of the class representatives may be calculated via the power map and the additional assumption in the previous lemma is satisfied. Of course the same holds if the prime p divides $|G|$ only with the first power.
In general the character table without the headline does not determine the order of the representatives of all classes. By 3.1 (iii) however one can decide which primes divide the order of arepresentative.

For the convenience of the reader we recall the following notions given in [12].

Definition 3.8. Let G and H be finite groups.

- A class structure of G is a labelled poset given by its normal subsets such that the label contains at least the size of the corresponding subset.
- A class structure X is called of type JH if the labels give the information which elements of X are normal subgroups and which are of the form $N \cdot C$ for some conjugacy class C of G and $N \trianglelefteq G$.
  X is called of type JHS if it is of type JH and additionally the labels indicate for all primes p which conjugacy classes contain elements of order a power of p.
  X is called of type JHB if X is of type JH and the labels contain the power map on the conjugacy classes.
- A bijection $\tau : G \to H$ is called a class correspondence if it gives a bijection on the conjugacy classes of G and H.
  τ is called of type JH if it is a bijection on the normal subgroups of G and H and if $\tau(N \cdot C) = \tau(N) \cdot \tau(C)$ for each normal subgroup N and each conjugacy class of G.
  τ is called of type JHS if it is of type JH and $\tau(x)$ is a p - element if, and only if, x is a p - element for each prime p.
  τ is called of type JHB if it is of type JH and $\tau(C)^n = \tau(C^n)$ for each $n \in \mathrm{N}$ and all conjugacy classes C of G.

Remarks. That JH comes from Jordan-Hölder, JHS from Jordan-Hölder-Sylow and JHB from Jordan-Hölder-Brauer is explained in [12, §1] as well as the following relationship with spectral and character tables.

- If $\mathrm{Spec}(G) = \mathrm{Spec}(H)$, then G and H are in class correspondence of type JHB.
- If $\mathrm{X}(G) = \mathrm{X}(H)$, then G and H are in class correspondence of type JHS.
- If G and H are in class correspondence of type JHB, then G and H are in G and H are in class correspondence of type JHS.

In terms of class structures (cf. [12]) Lemma 3.7 may be strengthened. Obviously a class structure on G of type JHB determines $n_p(G)$ provided G has cyclic Sylow p - subgroups. But in this case the situation is even better.

Proposition 3.9. *Let G be a finite group and suppose that G has cyclic Sylow p - subgroups.*

(i) *Then the order of the representatives of the conjugacy classes of p - elements in $X(G)$ is determined by $\mathrm{X}(G)$ provided G is p - soluble.*

(ii) *If H is a group which is in class correspondence of type JH to G then H has also cyclic Sylow p - subgroups.*

Proof. (i) $\mathrm{X}(G)$ determines $\mathrm{X}(G/N)$ for each normal subgroup N in a way that the induced map on the conjugacy classes of G is given. Therefore we can reduce by $O_{p'}(G)$ and may suppose that $O_{p'}(G) = 1$.

As G is p-soluble, the Fitting subgroup of G is a p - group and because $O_{p'}(G) = 1$ we have that $\mathrm{C}_G(F(G)) \leq F(G)$. As the Sylow p - subgroups are cyclic it follows that G has a normal cyclic Sylow p - subgroup P. By Lemma 3.1 (ii) $\mathrm{X}(G)$ shows precisely which conjugacy classes belong to normal subgroups of G sitting inside of P. Because P is cyclic P has for each divisor d of its order precisely one subgroup of order d. Thus we find the conjugacy classes having a generator of P as representative and part (i) is proved.

Suppose that G and H are in class correspondence of type JH. Suppose that G is p-soluble. Using [12, Lemma 1.8] we may assume that $O_{p'}(G) = O_{p'}(H) = 1$. The normal subgroup correspondence given by a class correspondence of type JH shows that H has a normal Sylow p - subgroup P^* which has for each divisor d of its order precisely one normal subgroup of order d.

We claim that this implies that P^* is cyclic. For $M := P^*/\Phi(P^*)$ is a semisimple $F_p(G/P^*)$ - module because $|G/P^*|$ and $|P^*|$ are coprime. By the normal subgroup structure of P^* we see that M has to be of order p. So $\Phi(P^*)$ has index p in P^* and it follows that P^* is cyclic. This proves part (ii) in the p - soluble case.

Suppose now that G is not p - soluble. We still can assume that $O_{p'}(G) = 1$. As G is not p - soluble it follows from a theorem of R. Brauer [2, Theorem 3C] that for a normal subgroup N of G either $P \subseteq N$ or $P \subseteq G/N$ for some $P \in \mathrm{Syl}_p(G)$. As $O_{p'}(G) = 1$ we obtain $P < N$. Thus G has precisely one minimal normal subgroup N. Since G is not p - soluble and p divides precisely one chief factor by Brauer's theorem we get that the generalized Fitting subgroup $F^*(G)$ is a simple nonabelian group S and it follows that G is an almost simple group of type S [1]. Moreover p does not divide $|G/S|$. A class structure of type JH determines the chief factors [12, Theorem 5]. Thus S is determined up to isomorphism and a group H in class correspondence of type JH to G must be an almost simple group of type S such that p does not divide $|H/S|$. Consquently H has cyclic Sylow p - subgroups as well. So part (ii) is completely established. □

If G has cyclic Sylow p - subgroups then the order of the centralizer of a Sylow p - subgroup P coincides with the smallest order of the centralizer of a p - element. Thus $\mathrm{X}(G)$ or even a class structure of type JHS of G determine $|\mathrm{C}_G(P)|$. The inertia group of the principal p - block of $\mathrm{C}_G(P)$ in $\mathrm{N}_G(P)$ is $\mathrm{N}_G(P)$. Let B_0 be the principal p - block of G. Then the celebrated Brauer - Dade theory of cyclic blocks shows that $|\mathrm{N}_G(P) : \mathrm{C}_G(P)|$ coincides with the number of non-exceptional characters in B_0 [6]. Consequently $\mathrm{X}(G)$ determines $n_p(G)$. We show that even a class structure of type JHS determines $n_p(G)$ provided G has cyclic Sylow p - subgroups.

Theorem 3.10. *Suppose that G has a cyclic Sylow p - subgroup and let H be a group such that G and H are in class correspondence of type JHS. Then $n_p(G) = n_p(H)$. Moreover $n_p(G)$ may be calculated from data given by a JHS - class structure.*

Proof. By Proposition 3.9(ii) we know that H has as well a cyclic Sylow p - subgroup. The result is clear when the Sylow p - subgroup is central. Thus we assume that $|\mathrm{C}_G(P)| < |G|$. Let h

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[1] If S is a finite non-abelian simple groups then we call a group $A(S)$ sandwiched between Inn$S$ and Aut$S$, i.e.

$$\mathrm{Inn}(S) \leq A(S) \leq \mathrm{Aut}(S),$$

an almost simple group of type S.

be a generator of a cyclic Sylow p - subgroup P. Certainly for each subgroup U of P we have that

$$\mathrm{C}_G(P) \subseteq \mathrm{C}_G(U) \text{ and } \mathrm{N}_G(P) \subseteq \mathrm{N}_G(U).$$

Let $U = \langle h^k \rangle$. Moreover $\mathrm{N}_G(P) < \mathrm{N}_G(U)$ if, and only if, $\mathrm{C}_G(h) < \mathrm{C}_G(h^k)$. Consequently subgroups generated by p - elements with minimal centralizer order have normalizers of the same order. Let M be the number of conjugacy classes of p - elements g such that $|\mathrm{C}_G(g)|$ is minimal. Note that a class correspondence of type JHS determines M. Because Sylow p - subgroups are cyclic we see that there is a certain $n \in \mathbb{N}_0$ such that all p - elements of order $> p^n$ have centralizers of minimal length. The number of such p - elements inside of P is $p^a - p^n$ if $|P| = p^a$. Thus we get

$$p^a - p^n = M \cdot \mathrm{N}_G(P)/\mathrm{C}_G(P).$$

The number of all such p - elements in G is $M \cdot |G/\mathrm{C}_G(P)|$. Note that a p - element with centralizer of minimal order cannot be contained in different Sylow p - subgroups. Consequently

$$n_p(G) = M \cdot \frac{L}{p^a - p^n},$$

where $L = |G|/|\mathrm{C}_G(g)|$ and g is a p - element with minimal centralizer order. M, L and p^a are given a class structure of type JHS. By Sylow's theorem $n_p(G) \equiv 1 \mod p$. Thus there is precisely one $n \in \mathbb{N}_0$ such that this congruence holds and $n_p(G)$ is determined by $\mathrm{X}(G)$. □

Looking for possible candidates of soluble groups whose Sylow numbers are not determined by their character table the following result shows that they must have at least with respect to two different primes non-cyclic Sylow subgroups.

Proposition 3.11. *Let G be a group of order $q^a \cdot p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$, where p_i are pairwise different primes and q is a prime different from all p_i. Assume further that all Sylow p_i - groups of G are cyclic. Then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof. Assume first that G is soluble. Let G be a counterexample of minimal order. Note that the result holds when G is a p - group. If N is a normal subgroup of G then G/N suffices the hypothesis of the theorem as well. So we may assume that the result holds for G/N. Assume that N is a minimal normal subgroup of G and that G is not simple. If N is not a q - group then N is cyclic and $|N| = p_i^k$ for some i and $k \leq a_i$. By Lemma 3.3 we get that $\mathrm{sn}(G)$ is determined by $\mathrm{X}(G)$. If N is a q - group then $n_q(G) = n_q(G/N)$ and $n_{p_i}(G)$ is determined by Theorem 3.9.

Assume now that G is insoluble. By the Feit - Thompson theorem 2 divides $|G|$ and it follows by [9, IV, Satz 2.8] that $q = 2$. As in the soluble case it follows that a minimal counterexample does not have minimal normal subgroups which are cyclic or of order 2^m.

Thus the generalized Fitting subgroup $F^*(G)$ only consists of the layer $E(G)$. As $\mathrm{C}(G) \leq \mathrm{C}_G(F^*(G)) \leq F(G) = 1$ we see that $E(G)$ is a simple non-abelian group S. Then G is isomorphic to an almost simple group of type S. By [1], see also [10, p.190], the only simple groups all of whose Sylow subgroups of odd order are cyclic are $\mathrm{PSL}(2, 2^f)$, $f > 1$, $\mathrm{PSL}(2, p)$, $p > 3$, $\mathrm{Sz}(2^{2n+1})$ and the Janko group J_1 of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$.
All these simple groups have a cyclic outer automorphism group. Thus for each divisor m of $|\mathrm{Out}(S)|$ there is precisely one almost simple group of type S of order $m \cdot |S|$. By 3.1(ii) and [14] $X(G)$ determines G up to isomorphism and in particular $\mathrm{sn}(G)$. □

Corollary 3.12. *Let G be a Frobenius group. Then $\mathrm{X}(G)$ determines $\mathrm{sn}(G)$.*

Proof. By Corollary 2.5 it suffices to show that the Sylow numbers of a Frobenius complement H are determined. Denote by K the Frobenius kernel then $\mathrm{X}(G)$ determines $\mathrm{X}(G/K) = \mathrm{X}(H)$. But Sylow subgroups of odd order of H are cyclic. Thus H satisfies the hypothesis of Proposition 3.11 and the result follows. □

4. Integral Group Rings

First we collect known results on properties of G determined by $\mathbb{Z}G$.

4.1. *Let G be a finite group.*

(i) *It is a result due to G. Glauberman that* $\mathbb{Z}G \cong \mathbb{Z}H$ *implicates that* $\mathrm{X}(G) = \mathrm{X}(H)$ *[11, 3.17]. Thus G and H share all properties given in 3.1.*
(ii) *If* $\mathbb{Z}G \cong \mathbb{Z}H$ *then there is a bijection* $\sigma : G \to H$ *such that the conjugacy classes of* $\sigma(g)$ *and g have the same length and representatives of the same order. Even the power map on the classes is determined.*
(iii) *Let* $N \trianglelefteq G$ *and* $\mathbb{Z}G \cong \mathbb{Z}H$*. Then exists* $M \trianglelefteq H$ *such that* $\mathbb{Z}G/N \cong \mathbb{Z}H/M$*. If* $g \in N$ *then* $\sigma(g) \in M$*, where* σ *is the bijection in (ii).*
(iv) F^* **- theorem.** *Assume that the generalized Fitting subgroup* $F^*(G/O_{p'}(G))$ *is a p - group (i.e. that G is p - constrained) then* $\mathbb{Z}G \cong \mathbb{Z}H$ *implies that* $G \cong H$ *[19].*

Theorem 4.2. *Let G be a finite soluble group and assume that* $\mathbb{Z}G \cong \mathbb{Z}H$*. Then* $\mathrm{sn}(G) = \mathrm{sn}(H)$*.*

Proof: If $|G| = p^k$, the result holds. Assume $|G| = p^a q^b$. By Theorem 2.1 we get

$$\begin{aligned} n_p(G) &= n_p(O_{q'}(G))n_p(G/O_{q'}(G))n_p(\mathrm{N}_{PO_{q'}(G)}(P \cap O_{q'}(G))/(P \cap O_{q'}(G)) \\ &= n_p(G/O_{q'}(G)). \end{aligned}$$

As $O_{q'}(G/O_{q'}(G)) = 1$, we conclude by the F^*-Theorem, that $G/O_{q'} \cong H/O_{q'}(H)$ and therefore $n_p(G) = n_p(H)$. For $n_q(G) = n_q(H)$ consider analogously $G/O_{p'}(G)$.

Assume $|\pi(G)| \geq 3$. If $O_{p'}(G) = 1$ for any $p \in \pi(G)$ the proposition follows immediately by the F^*-Theorem. So we assume that $O_{p'}(G) \neq 1$ for each $p \in \pi(G)$.
Let $p \in \pi(G)$ be fixed. If $O_p(G) \neq 1$, then we can consider $\mathbb{Z}G/O_p(G)$ without altering the Sylow p-number. Note that by 4.1(i) and 3.1(ii) it follows that $\mathbb{Z}G/O_p(G) \cong \mathbb{Z}H/O_p(H)$. Thus by induction on the order of G we get that $n_p(G) = n_P(H)$.

Without loss of generality we can assume that $O_p(G) = 1$. As $O_{s'}(G) \neq 1$ for each $s \in \pi(G)$ and because G is soluble, there exist $q, r \in \pi(G)\backslash\{p\}$ such that $O_q(G) \neq 1$ and $O_r(G) \neq 1$. Hall's Theorem 2.1 yields (with normal subgroup $O_q(G)$ and $P \in \mathrm{Syl}_p(G)$)

$$n_p(G) = n_p(O_q(G)) \cdot n_p(G/O_q(G)) \cdot n_p(PO_q(G)) = n_p(G/O_q(G)) \cdot n_p(PO_q(G)).$$

As before we get by induction that $n_p(G/O_q(G)) = n_p(H/O_q(H))$. Instead of $P \in \mathrm{Syl}_p(G)$ we consider in $\overline{G} := G/O_r(G)$ the Sylow group $\overline{P} = PO_r(G)/O_r(G) \in \mathrm{Syl}_p(\overline{G})$. Denote by $\overline{O_q(G)}$ the image of $O_q(G)$ in $\overline{G}$. Then again by Theorem 2.1

$$n_p(\bar{G}) = n_p(\overline{G}/\overline{O_q(G)}) \cdot n_p(\overline{P}\overline{O_q(G)}).$$

With respect ot H - using the bar notation as well for reduction modulo $O_r(H)$ - we get for a Sylow subgroup $\tilde{P} \in \mathrm{Syl}_p(H)$

$$n_p(\overline{H}) = n_p(\overline{H}/\overline{O_q(H)}) \cdot n_p(\overline{\tilde{P}}\overline{O_q(H)}).$$

By 4.1(i) and 3.1(ii) and induction it follows that

$$n_p(\overline{G}) = n_p(\overline{H}) \quad \text{and} \quad n_p(\overline{G}/\overline{O_q(G)} = n_p(\overline{H}/\overline{O_q(H)}.$$

Thus it suffices to show that $n_p(PO_q(G)) = n_p(\overline{P}\overline{O_q(G)})$. Denote by $\overline{x} \in \overline{G}$ the corresponding element of $x \in G$.
By Proposition 2.4 we obtain that $n_p(PO_q(G)) = |O_q(G) : \mathrm{C}_G(P) \cap O_q(G)|$. If $x \in O_q(G)$ centralizes P, then $\overline{x}$ centralizes $\overline{P}$. Assume there exists $x \in O_q(G)$, such that $\overline{x} \in \mathrm{C}_{\overline{G}}(\overline{P})$, but $x \notin \mathrm{C}_G(P)$. As x does not centralize P, there exists $w \in P\backslash\{1\}$ with $x^{-1}wx = wm$ for some $m \in O_r(G)\backslash\{1\}$. Thus $w^{-1}x^{-1}w = mx^{-1}$. But the left hand side lies in the normal subgroup $O_q(G)$ whileas the right hand side does not because $m \neq 1$. This contradiction completes the proof □

$$x^{-2}wx^2 = x^{-1}wmx = x^{-1}wxm = wm^2.$$

Remark. The arguments used in the proof above may be generalized to insoluble groups.

Finally we show results for special classes of insoluble groups with abelian Sylow 2 - subgroups.

Theorem 4.3.

a) *$\mathbb{Z}G$ determines $n_2(G)$ provided G has abelian Sylow 2 - subgroups.*
b) *$\mathbb{Z}G$ determines $\mathrm{sn}(G)$ provided G has an abelian Sylow 2 - subgroup of order ≤ 8.*

Proof. a) If G is a finite group with abelian Sylow 2 - subgroup then G has a normal series

$$1 < M < N < G$$

such that $M = O_{2'}(G)$, G/N has odd order and N/M is a direct product of simple groups with abelian Sylow 2 - subgroups and an abelian 2 - group [21]. Again we consider a counterexample of minimal order.
Suppose that G has a minimal normal subgroup V which is not soluble. Clearly $n_2(G) = n_2(N)$. Then V is a normal subgroup of N with $\mathrm{C}_N(V) \cap V = 1$. Let $g \in N$. Then conjugation with g induces an inner automorphism of V.

Thus $N = \mathrm{C}_N(V) \cdot V$. It follows that N is a direct product of the form $V \times \mathrm{C}_N(V)$ and $n_2(N) = n_2(V) \times n_2(\mathrm{C}_N(V)$. Moreover $n_2(\mathrm{C}_N(V) = n_2(G/V)$. Using 4.1 V is determined up to isomorphism by $\mathbb{Z}G$ and G/V is not a counterexample. Thus we may assume that a minimal counterexample does not have an insoluble minimal normal subgroup.

If $M = 1$ then N has to be a 2 - subgroup and G has a normal Sylow 2 - subgroup.

Assume that $M \neq 1$ and that G contains two minimal normal soluble subgroups of coprime order then by Proposition 2.7we see that G is not a minimal counterexample.

Finally, if G has only minimal normal subgroups which are q - groups for some prime q then $\mathrm{soc}(G)$ must be contained in M if $M \neq 1$. Moreover the Fitting subgroup $F(G)$ is a q - group. Let $C = \mathrm{C}_G(F(G))$. Clearly C is normal in G.

If C is not soluble then consider its layer $E(C)$. The components of $E(C)$ involve simple groups with abelian Sylow 2 - subgroups. All these simple groups do not have a Schur multiplier involving odd primes. So G has an insoluble minimal normal subgroup.

Thus C is soluble. Its Fitting subgroup $F(C)$ is normal in G and must be a q - group. It follows that $C \subset F(G)$ and G is q - constrained. By the F^* - theorem 4.1 we see hat $\mathbb{Z}G$ determines G up to isomorphism. Thus G is not a counterexample.

b) We argue as in part a) till the stage that $M = 1$. Because by assumption the abelian Sylow 2 - subgroup has order at most 2^3 it follows that N is either a direct product of a group of order 2 and a non-abelian simple group with Kleinian fourgroups as Sylow 2 - subgroups or it is simple. In the first case G is a direct product of C_2 with an almost simple group of type S and S is isomorphic to a $\mathrm{PSL}(2,p)$. In the second case G is almost simple of type S with $S = \mathrm{J}_1$ or a Ree group. All these almost simple groups are determined by their integral group rings and therefore in both cases G as well. □

References

[1] M. Aschbacher, Thin finite simple groups, Journal of Algebra **54**, 50-152 (1978)
[2] R. Brauer, On finite groups with cyclic Sylow subgroups I, J. of Alg. **40** (1976), 556 – 584
[3] N. Chigira, Number of Sylow subgroups and p-Nilpotence of finite groups, Journal of Algebra **201** (1998), 71 – 85
[4] W. Guo, Finite Groups with given indices of normalizers of Sylow subgroups, Siberian Math. Journal 37 No. **2** (1996), 253 – 257
[5] M. Hall, Jr., On the Number of Sylow Subgroups in a Finite Group, Journal of Algebra **7** (1967), 363 – 371
[6] C. E. Dade, Blocks with cyclic defect groups, Ann. of Math. **84** (1966), 20 – 48
[7] M. Hertweck, A counterexample to the isomorphism problem for integral group rings, Ann. of Math. **154** (2001), 115 – 138
[8] M. Hertweck and W. Kimmerle, On principal blocks of p - constrained groups, Proc. LMS (3) **84**, no. 1, Groups St. Andrews (2002), 179 – 193
[9] B. Huppert, Endliche Gruppen, Grundlehren der math. Wiss. Band **134**, Springer-Verlag Berlin - Heidelberg (1967)
[10] B. Huppert and N. Blackburn, Finite Groups III, Grundlehren der math. Wiss. Band **243**, Springer-Verlag Berlin Heidelberg New York (1982)

[11] M. Isaacs, Character theory of finite groups, Academic Press. New York (1976)
[12] W. Kimmerle and R. Sandling, Group theoretic and group ring theoretic determination of certain Sylow and Hall subgroups and the resolution of a question of R. Brauer. Journal of Algebra **171**, 329-346 (1995)
[13] W. Kimmerle, Beiträge zur ganzzahligen Darstellungstheorie endlicher Gruppen, Bayr.Math.Schriften Heft 36 (1991)
[14] W. Kimmerle, R. Lyons, R. Sandling and D. Teague, Composition factors from the group ring and Artin's theorem on the orders of finite simple groups, Proc.London Math.Soc.(3) **60** (1990), 89 – 122
[15] I. Köster, Sylowzahlen, Master Thesis (2013), University of Stuttgart (2013)
[16] X. Li, Arithmetic Properties and the Characterization of Finite Groups, Dissertation an der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Potsdam (2006)
[17] F. Luca, Groups with two Sylow numbers are solvable, Arch. Math. **71** (1998), 95 – 96
[18] A. Moretó , Groups with two Sylow numbers are the product of two nilpotent Hall subgroups, Arch. Math. **99** (2012), 301 – 304
[19] K. W. Roggenkamp and L. L. Scott, A strong answer to the isomorphism for finite p - solvable groups, Manuscript Stuttgart (1987)
[20] S. K. Sehgal, Units in integral group rings, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Essex (1993)
[21] J. H. Walter, The characterization of finite groups with Abelian Sylow 2 - subgroups, Ann. of Math. **89**, 405 - 514 (1969)
[22] Jiping Zhang, Sylow Numbers of Finite Groups, Journal of Algebra **176** (1995), 111 – 123

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