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Cubic Spline Approximation of a Circle with Maximal Smoothness and Accuracy

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Abstract

We construct cubic spline approximations of a circle which are four times continuously differentiable and converge with order six.

Keywords. Bézier curve, cubic spline, geometric smoothness, high accuracy approximation of circles

Applying concepts of differential geometry to CAGD has led to a number of interesting results. An example is the surprising fact that spline curves can approximate with higher order than spline functions. In addition, smoothness constraints become less restrictive. A general statement to this effect is the following

Conjecture. For smooth curves in $\mathbb{R}^d$, which satisfy mild generic assumptions, there exist spline approximations of degree $\le n$ which are $\alpha = n - 1 + \lfloor (n-1)/(d-1) \rfloor$ times continuously differentiable and converge with order $\alpha + 2$.

For low degree, many schemes of optimal order have been proposed (cf. [3, 7] and the references given there for an overview). However, except for the simplest case $n = d = 2$ treated in [6], the gain in accuracy has not yet been combined with a possible improvement of smoothness. This motivated this short note, which, aside from some obvious practical relevance, supports the conjecture for a canonical test case.

The construction of a, in some sense optimal, cubic spline approximation $t \mapsto p(t)$ of a circle is straightforward. We interpolate the circle at the points

$$P_k = (\cos(k\varphi),\, \sin(k\varphi)), \quad \varphi = 2\pi/m\,,$$

which separate the m cubic Bézier segments of p, and choose $k = 0, 1, \ldots, m-1$ as knots. Morerover, we match the tangents at P_k. For a completely symmetric configuration, this leaves the length of the tangent vectors as the only degree of freedom. In terms of a corresponding parameter δ, the interior control points of the curve segments are

$$P_k^{\pm} = P_k \pm \delta(-\sin(k\varphi),\, \cos(k\varphi)), \quad \delta = |p'(k)|/3\,.$$

In particular, as depicted in the figure,

$$(1,\,0), \quad (1,\delta), \quad (\cos\varphi,\, \sin\varphi) - \delta(-\sin\varphi,\, \cos\varphi), \quad (\cos\varphi,\, \sin\varphi)$$

are the control points for the Bézier segment from P_0 to P_1.

Each Bézier segment has the curvature

$$\kappa = \frac{2(1 - \cos\varphi - \delta\sin\varphi)}{3\delta^2}$$

at its endpoints. Choosing the parameter δ to match the curvature of the circle leads to the geometric Hermite interpolant [2]. While this approximation is sixth order accurate as $\varphi \to 0$, it merely has the standard smoothness $\alpha - 2 = 2$. Several other constructions have been proposed [4, 5, 1], focussing on minimizing the error while maintaining standard smoothness. The alternative, considered in this note, is to achieve the maximal order of differentiability $\alpha = 4$. To this end, we exploit the well known fact that, for geometric smoothness, the parametrization needs not to be smooth. For our purposes, the following definition is most convenient.

Geometric Smoothness. Identifying a planar curve p in a neighborhood of a point P_0 with the graph of a function f, the smoothness of p and f at P_0 are equivalent.

We apply this criterion to the spline $t \mapsto p(t) = (x(t),\, y(t))$ in a neighborhood of $P_0 = (1, 0)$. Since the tangent at P_0 is vertical,

$$x(t) = f(y(t)), \quad t \approx 0\,.$$

By symmetry, f is an even function ($f(y) = f(-y)$), implying that all its even derivatives are continuous at $y(0) = 0$. Moreover, since by construction

$$p'(0) = (x'(0),\, y'(0)) = (0, 3\delta)$$

is the common tangent vector of the cubic segments, joining at $t = 0$, $f'(0)$ is zero. We will now choose δ so that

$$f'''(0^+) = 0\,,$$

which, by symmetry, implies $f'''(0^-) = 0$, and thus yields a four times continuously differentiable spline curve; f'''' is continuous by symmetry, as remarked above. By the chain rule,

$$\begin{aligned} x' &= f'(y)y' \\ x'' &= f''(y)(y')^2 + f'(y)y'' \\ x''' &= f'''(y)(y')^3 + 3f''(y)y''y' + f'(y)y'''\,. \end{aligned}$$

Substituting $y(0) = 0$ and $x'(0) = 0$ yields

$$f'(0) = 0, \quad f''(0) = x''(0)/(y'(0))^2$$

and, by the third equation, $f'''(0^+) = 0$ is equivalent to

$$x'''(0)\, y'(0) = 3x''(0)y''(0)\,.$$

Expressing the derivatives in terms of the components of the Bézier control points, we obtain a quadratic equation for the parameter δ:

$$3\sin\varphi(3 + 2\cos\varphi)\delta^2 + 4(3\cos^2\varphi + \cos\varphi - 4)\delta + 6\sin\varphi(1 - \cos\varphi) = 0\,.$$

Both solutions

$$\delta_\pm = \frac{1 - \cos\varphi}{3\sin\varphi(3 + 2\cos\varphi)} \left(2(4 + 3\cos\varphi) \pm \sqrt{2(5 + 3\cos\varphi)}\right)$$

yield C^4 cubic spline curves.

We now show that the spline curve corresponding to $\delta = \delta_-$ not only has maximal smoothness $\alpha = 4$, but is also sixth order accurate. The other solution δ_+ merely yields the standard accuracy and will not be considered in the sequel. By symmetry, and in view of the Hermite interpolation conditions at the endpoints,

$$e(t) := 1 - x(t)^2 - y(t)^2 = t^3(1 - t)^3 g(\varphi) + t^2(1 - t)^2 h(\varphi)\,,$$

where computer algebra provides the explicit expressions

$$\begin{aligned} g(\varphi) &= 18(1 + \cos\varphi)\delta^2 - 24\delta\sin\varphi + 8(1 - \cos\varphi), \\ h(\varphi) &= -9\delta^2 - 6\delta\sin\varphi + 6(1 - \cos\varphi)\,. \end{aligned}$$

It is not difficult to show that e is positive on $[0, 1]$ with a unique maximum at $t = 1/2$:

$$\begin{aligned} E := & \max_{0 \le t \le 1} e(t) = e(1/2) = \frac{9}{32}(\cos\varphi - 1)\delta^2 - \frac{3}{4}\delta\sin\varphi + \frac{1}{2} - \frac{1}{2}\cos\varphi \\ = & \frac{3}{4096}\varphi^6 + O(\varphi^8)\,. \end{aligned}$$

In a final step, we can reduce the one-sided error by scaling the parametrization p to obtain $2m$ equal oscillations with radii $1 \pm E_\star$. Observing that, by definition of $e(t)$, the distance of $p(t)$ from the origin varies from $\sqrt{1 - E}$ to 1, we define

$$p_\star = \frac{2}{1 + \sqrt{1 - E}}\, p$$

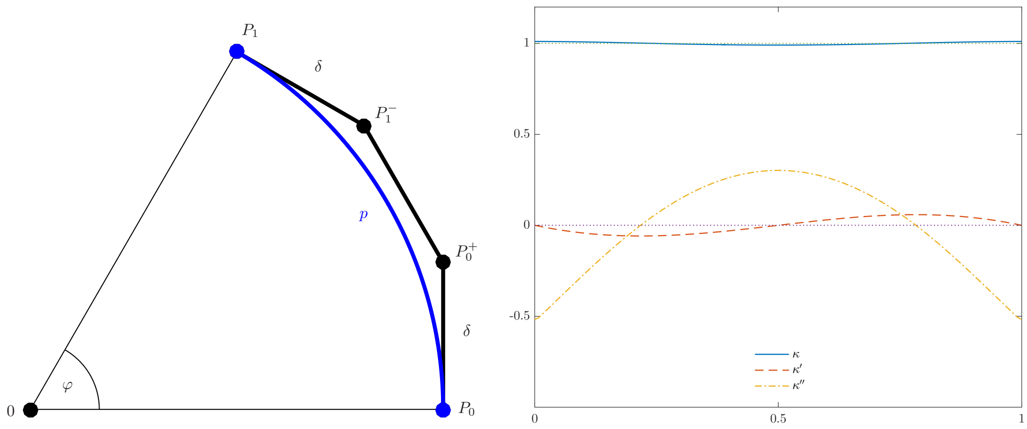

Fig. Bézier segment with control points (left), graphs of κ, κ' and κ'' (right)

as our optimal spline approximation with the error

$$E_\star = \frac{1-\sqrt{1-E}}{1+\sqrt{1-E}} = \frac{3}{16384}\varphi^6 + O(\varphi^8)\,.$$

The elementary example presented in this note motivates the construction of general spline curves with maximal smoothness from control polygons. While, in view of the nonlinearity of the geometric smoothness constraints analytical solutions seem out of reach, with todays computing resources, numerical techniques are quite promising. We think that it is worthwhile to pursue this in the future.

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