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Abstract

We characterize the Miquelian Laguerre planes of odd order by the existence of shift groups in affine derivations.

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Introduction

A finite Laguerre plane $\mathscr{L} = (P, \mathscr{C}, \mathscr{G})$ of order n consists of a set P of $n(n+1)$ points, a set $\mathscr{C}$ of n^3 circles and a set $\mathscr{G}$ of $n+1$ generators, where both circles and generators are subsets of P, such that the following three axioms are satisfied.

(G) $\mathscr{G}$ partitions P, each generator contains n points, and there are $n+1$ generators.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator are joined by a unique circle.

Circles through x are called *touching in x* if they are equal or have no other point in common. The set of all circles through a given point x is denoted by $\mathscr{C}_x$. The *derived affine plane* $\mathbb{A}_x = (P \smallsetminus [x], \mathscr{C}_x \cup \mathscr{G} \smallsetminus \{[x]\})$ at a point $x \in P$ has the collection of all points not on the generator $[x]$ through x as point set and, as lines, all circles passing through x (without the point x) and all generators apart from $[x]$. The axioms above easily yield that $\mathbb{A}_x$ is an affine plane, indeed. We refer to the generators as *vertical lines* in $\mathbb{A}_x$. Circles that touch each other in x give parallel lines in $\mathbb{A}_x$. A line W is introduced to obtain the projective completion $\mathbb{P}_x$ of $\mathbb{A}_x$; the common point of the verticals will be denoted by $v \in W$.

The group $\mathrm{Aut}(\mathscr{L})$ of all automorphisms of a Laguerre plane $\mathscr{L}$ acts on the set $\mathscr{G}$ of generators. We call $\mathscr{L}$ an *elation Laguerre plane* if the kernel Δ of that action acts transitively on the set $\mathscr{C}$ of circles. It is known (see [5, 1.3]) that in every finite elation Laguerre plane the group Δ has a (unique) regular normal subgroup E called the *elation group*. For more details on elation Laguerre planes, we refer the reader to the introduction in [6].

In the present note, we only use a weaker transitivity assumption on Δ but combine this with additional assumptions. Our results can (and will) be applied to elation Laguerre planes with additional homogeneity assumptions, e.g. in [7] (cf. 2.3 below).

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1 Translation planes

1.1 Theorem. *Let $\mathbb{P}$ be a finite projective plane of order n. Assume that a subgroup $D \le \mathrm{Aut}(\mathbb{P})$ fixes each point on some line L. If n^2 divides the order of D then D contains a subgroup T of order n^2 consisting of elations with axis L. In particular, the plane $\mathbb{P}$ is a translation plane, and the order n is a prime power.*

Proof. For each non-trivial element $\delta \in D$ there is a (unique) center c_δ; i.e. a point c_δ such that δ fixes each line through c_δ ([1], see [2, Thm. 4.9]). The elations in D are just those in the set $T := \{\mathrm{id}\} \cup \left\{\tau \in D \smallsetminus \{\mathrm{id}\} \mid c_\tau \in L\right\}$; that set forms a normal subgroup of D (see [2, Thm 4.13]).

For any point x outside L, the stabilizer D_x consists of id and elements with center x. The order of any element of D_x divides $n-1$. So the order of D_x and the number n^2 of points outside L are co-prime, and D acts transitively on the set A of points outside L. For each $\delta \in D \smallsetminus T$ we have $c_\delta \notin L$, and $\delta \in D_{c_\delta}$ yields that the order of δ divides $n-1$, and is co-prime to n^2.

Let $\mathscr{B}$ denote the set of T-orbits in A. Then D acts on $\mathscr{B}$, and so does D/T because $T \trianglelefteq D$ acts trivially on $\mathscr{B}$. Transitivity of D on A implies that D/T is transitive on $\mathscr{B}$. Now $|\mathscr{B}| = n^2/|T|$ divides $|D/T|$. The latter order is co-prime to n^2 because each member of the quotient has a representative of order co-prime to n^2. So $|\mathscr{B}| = 1$, and transitivity of T is proved. □

1.2 Theorem. *Let $\mathscr{L}$ be a Laguerre plane of finite order n. If ∞ is a point such that n^2 divides the order of the stabilizer Δ_∞ then the derived projective plane $\mathbb{P}_\infty$ is a dual translation plane, and the order n is a prime power.*

Proof. The group D induced by Δ_∞ on the dual $\mathbb{P}$ of $\mathbb{P}_\infty$ satisfies the assumptions of 1.1. □

1.3 Theorem. *Let $\mathscr{L}$ be a Laguerre plane of finite order n, and assume that there is a point ∞ such that n^2 divides the order of the stabilizer Δ_∞. If there exist a circle $K \in \mathscr{C}_\infty$ and a subgroup $H \le \mathrm{Aut}(\mathscr{L})_\infty$ such that H fixes each circle touching K in ∞ and H acts transitively on $K \smallsetminus \{\infty\}$, then $\mathbb{P}_\infty$ has Lenz type V (at least), and is coordinatized by a semifield.*

Proof. From 1.2 we know that $\mathbb{P}_\infty$ is a dual translation plane. The translation axis in the dual of $\mathbb{P}_\infty$ is the common point v for the generators in the projective closure of $\mathbb{A}_\infty$. The elations of $\mathbb{P}_\infty$ with center v and axis W form a group of order n; we denote that group by V and note that V is a group of translations of $\mathbb{A}_\infty$.

Our assumptions on H secure that H induces a group of translations of $\mathbb{A}_\infty$; the common center is the point at infinity for the "horizontal line" $K \smallsetminus \{\infty\}$. We obtain a transitive group HV of translations on $\mathbb{A}_\infty$. So $\mathbb{P}_\infty$ is also a translation plane, and has Lenz type V at least. □

2 Shift groups

Recall that a shift group on a projective plane is a group of automorphisms fixing an incident point-line pair (x, Y) and acting regularly both on the set of points outside Y and on the set of lines not through x.

2.1 Theorem. *Let $\mathscr{L}$ be a finite elation Laguerre plane of odd order, and assume that there exists a point u and a subgroup $S \le \mathrm{Aut}(\mathscr{L})_u$ such that S induces a transitive group of translations on the affine plane $\mathbb{A}_u$.*

1. *If $s \in [u] \smallsetminus \{u\}$ is fixed by S then S induces a shift group on $\mathbb{P}_s$.*
2. *If S fixes a point t of $\mathscr{L}$ and induces a transitive group of translations on $\mathbb{A}_t$ then $t = u$.*

Proof. Let n denote the order of $\mathscr{L}$. Assume that $s \in [u] \smallsetminus \{u\}$ is fixed by S. Then S induces a group of automorphisms of $\mathbb{P}_s$; we have to exhibit an incident point-line pair (x, Y) such that S acts regularly both on the set of points outside Y and on the set of lines not through x.

It is obvious that S acts regularly on the set of affine points in $\mathbb{P}_s$ because that set coincides with the set of points of $\mathbb{A}_u$. We let the line W at infinity play the role of Y. Also, the set of vertical lines (induced by generators) is invariant under S, we let their point at infinity play the role of x (so $x = v \in W$).

It remains to show that S acts regularly on the set of non-vertical lines of $\mathbb{A}_s$; these lines are induced by the circles through s. Assume that $\tau \in S$ fixes a circle C through s. Our assumption that n be odd implies that the translation of $\mathbb{A}_u$ induced by τ does not have any orbit of length 2, and we obtain that τ is trivial if there is a set of one or two points outside $[u]$ invariant under τ.

As τ induces a translation on $\mathbb{A}_u$, there exists $D \in \mathscr{C}_u$ such that τ fixes each circle touching D in u (these circles induce the parallels to the line induced by D on $\mathbb{A}_u$). Pick a point $z \in C$, and let D' be the circle through z touching D in u. Then τ leaves the intersection $D' \cap C$ invariant. This is a set with one ore two elements, and we find that τ is trivial. So the orbit of C under S has length $|S| = n^2$, and fills all of $\mathscr{C}_s$. Thus S acts regularly on the set of non-vertical lines of $\mathbb{A}_s$, as required.

Now assume that S fixes t and induces a transitive group of translations on $\mathbb{A}_t$. Then $t \in [u]$ because S acts regularly on the set of points outside $[u]$. For any circle $C \in \mathscr{C}_t$, we pick two points $a, b \in C \smallsetminus \{t\}$. Then there exists $\tau \in S$ such that $\tau(a) = b$. As τ is a translation both of $\mathbb{A}_u$ and of $\mathbb{A}_t$, the orbit of a under $\langle\tau\rangle$ is contained both in the line C of $\mathbb{A}_t$ and in some line B of $\mathbb{A}_u$, that is, in some circle B through u. Since n is odd, that orbit has at least three points, and $B = C$. This yields $t = u$, as claimed. □

2.2 Theorem. *Assume that $\mathscr{L}$ is a Laguerre plane of odd order n, and let ∞ be a point. Let U denote the set of all points $u \in [\infty] \smallsetminus \{\infty\}$ such that there exists a subgroup $S_u \le \mathrm{Aut}(\mathscr{L})$ of order n^2 fixing both ∞ and u and acting as a group of translations on $\mathbb{A}_u$. Then the following hold:*

1. *There are at least $|U|$ many different shift groups on $\mathbb{P}_\infty$.*
2. *If $|U| > 1$ then $\mathbb{A}_\infty$ is a translation plane.*
3. *If $\mathbb{A}_\infty$ is a translation plane and U is not empty then $\mathbb{P}_\infty$ has Lenz type V at least, can be coordinatized by a commutative semifield, and the middle nucleus of such a coordinatizing semifield has order at least $|U| + 1$.*
4. *If $|U| > \sqrt{n}$ then $\mathbb{P}_\infty$ is Desarguesian.*

Proof. Using 2.1 we see for any $u \in U$ that S_u is a shift group on $\mathbb{A}_\infty$, and different points $t, u \in U$ yield different groups S_t and S_u. This gives the first assertion. All these shift groups have the same fixed flag in $\mathbb{P}_\infty$.

If a finite projective plane admits more than one shift group, it is a translation plane, see [3, 10.2]. If a translation plane admits at least one shift group then it can be coordinatized by a commutative semifield ([3, 9.12], [4]) and the different shift groups with the same fixed flag are parameterized by the non-zero elements of the middle nucleus of such a semifield, see [3, 9.4].

The additive group of the coordinatizing semifield forms a vector space over the middle nucleus (see [2, p. 170]). If the middle nucleus has more than $\sqrt{n}$ elements then that vector space has dimension one, and the middle nucleus coincides with the semifield. This means that the semifield is a field, and the plane is Desarguesian. □

In [7], our present result 2.2 is used to prove the following:

2.3 Theorem. *Let $\mathscr{L}$ be an elation Laguerre plane of odd order. If there exists a point ∞ such that* $\mathrm{Aut}(\mathscr{L})_\infty$ *acts two-transitively on* $\mathscr{G} \smallsetminus \{[\infty]\}$ *then the affine plane* $\mathbb{A}_\infty$ *is Desarguesian, and* $\mathscr{L}$ *is Miquelian.* □

2.4 Remark. If $\mathbb{P}$ is a projective plane of even order then a shift group on $\mathbb{P}$ will never be elementary abelian, see [3, 1.5, 5.8]. Thus a shift group on such a plane will not act as transitive group of translations on any other affine plane (of the same order).

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