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AN ALTERNATIVE PROOF OF THE H^1-STABILITY OF THE L_2-PROJECTION ON GRADED MESHES

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Abstract. We show that the L_2-orthogonal projections onto the space of continuous Lagrange finite elements up to order four is H^1-stable for adaptive triangulations in 2d, which are generated by either Newest Vertex Bisection or Red-Blue-Green Refinement. We prove this by extending the techniques used in [3] and [4], to to higher polynomial order using properties of the generalized mesh-size function presented in [8]. We extend this result partially to Red-Green Refinement.

1. Introduction

The L_2-projection onto discrete spaces plays an essential role in the analysis of finite element discretizations. Tantardini and Veeser [14] have shown that the implicit Euler discretization in time of the heat equation leads to a discretely inf-sup stable bilinear form, provided that the L_2-projection onto the finite element space for the spacial discretization is H^1-stable and that for the semi-discretization in space the H^1-stability is *necessary and sufficient* for the discrete inf-sup condition. Heine, Gaspoz and Siebert have developed a variational formulation for Dirichlet boundary data suitable for optimal control problems with Dirichlet boundary control.

On uniform grids, H^1-stability of the L_2-projection can easily be deduced by an inverse estimate, using its definition and employing an H^1-stable interpolation operator. This simple proof cannot be transfered to adaptively generated meshes, where $h_{\max}$ and $h_{\min}$ are in general entirely unrelated. Moreover, the example in [1, §7] suggests that the L_2-projection is not H^1-stable if the local mesh-size changes too fast.

Since adaptive grids have become an important tool in science and engineering there has been an increase of interest in proving H^1-stability of the L_2-projection on graded meshes. By now, there are mainly four proofs in higher space dimension. Crouzeix and Thomée decompose a triangulation in 2d into rings of elements satisfying a suitable grading condition to show stability [7]. Bramble, Pasciak, Steinbach give in any dimension a condition on a disturbed element mass matrix, which is the basis of the stability proof [3]. This condition reduces for lowest order finite elements to a grading condition of a regularized mesh-size function; compare with (6.6) in [3]. Combining both approaches Carstensen was able to prove stability relying on weaker conditions [4]. In [9] grading estimates are produced for Newest Vertex Bisection and a proof of H^1-stability of the L_2-projection is given for linear piece-wise elements. The most recent result of Bank and Yserentant in 2d and 3d assumes a suitable decomposition of the grid induced by the level of elements [1]. This last technique was then used by [8] coupled with a sharp grading estimate to prove the H^1-stability of the L_2-projection on meshes generated by Newest Vertex

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Bisection from an initial mesh $\mathcal{T}_0$ satisfying a *reflected neighbours condition* (see Assumption 3.1).

Main Theorem. *The L_2-orthogonal projection onto the space of continuous Lagrange finite elements is H^1-stable up to order four for meshes generated by Newest Vertex Bisection or Red-Blue-Green Refinement.*

This result completes the stability result of [8] for the case p=2 which was still an open issue.

Throughout the article we use the notation $a \lesssim b$ for $a \le Cb$ with some generic constant C that solely depends on $\mathcal{T}_0$ and the polynomial degree p, and write $a \approx b$ whenever $a \lesssim b \lesssim a$.

2. H^1-Stability on Suitably Graded Meshes

We introduce notation related to triangulations and finite element spaces and then give an easy proof of H^1-stability of the L_2-orthogonal projection for triangulations satisfying a suitable grading. The discussion of the feasibility of such grading is postponed to the next section.

2.1. **Triangulation and finite element space.** Let Ω be a bounded polyhedral domain in $\mathbb{R}^d$ for $d \in \mathbb{N}$, and let $\mathcal{T}$ be a conforming, shape-regular and exact triangulation of Ω. We denote by $\mathcal{V}$ the set of all vertices of $\mathcal{T}$. For $T \in \mathcal{T}$ we set $\mathcal{V}(T) = \mathcal{V} \cap T$ and for $z \in \mathcal{V}$ we define $\mathcal{T}(z) = \{T \in \mathcal{T} \mid z \in T\}$. We use $\mathcal{E}$ for the set of all edges of $\mathcal{T}$. Finally, for $T \in \mathcal{T}$ we let $h_T = |T|^{1/2} \approx \operatorname{diam}(T)$, and denote by $h \in L_\infty(\Omega)$ the piecewise constant mesh-size function with $h_{|T} = h_T$, $T \in \mathcal{T}$.

We set $H^1_D(\Omega) = \{v \in H^1(\Omega) \mid v \equiv 0 \text{ on } \partial_D\Omega\}$, where $\partial_D\Omega \subset \partial\Omega$ is the Dirichlet boundary. We suppose that $\mathcal{T}$ meshes $\partial_D\Omega$ exactly, i. e., $\partial_D\Omega$ is the union of boundary sides. For fixed $p \in \mathbb{N}$ we then consider conforming, piecewise polynomials of degree p over $\mathcal{T}$, i. e.,

$$\mathbb{V} = \mathbb{V}(\mathcal{T}, p) = \{V \in H^1_D(\Omega) \mid V_{|T} \in \mathbb{P}_p,\ T \in \mathcal{T}\}.$$

We denote by $\Pi\colon L_2(\Omega) \to \mathbb{V}$ the L_2-orthogonal projection, which is characterized by

$$\langle \Pi u - u,\, V\rangle_\Omega := \int_\Omega (\Pi u - u)V = 0 \qquad \forall\, u \in L_2(\Omega),\ V \in \mathbb{V}.$$

We shall make use of the Scott-Zhang interpolant $I_{\mathsf{SZ}}\colon H^1_D(\Omega) \to \mathbb{V}$, which satisfies

$$\|h^{-1}(I_{\mathsf{SZ}}u - u)\|_\Omega + \|\nabla I_{\mathsf{SZ}}u\|_\Omega \lesssim \|\nabla u\|_\Omega \qquad \forall\, u \in H^1_D(\Omega). \tag{2.1}$$

The hidden constant solely depends on the shape regularity coefficient of $\mathcal{T}$; compare with [12]. We also utilize the Lagrange interpolant $I_{\mathsf{L}}\colon C^0(\bar\Omega) \cap H^1_D(\Omega) \to \mathbb{V}$, which is uniquely determined by its nodal values at the Lagrange nodes $\mathcal{N}$ of $\mathbb{V}$, this means, $(I_{\mathsf{L}}u) = \sum_{a\in\mathcal{N}} u(a)\Phi_a$, where $\{\Phi_a \mid a \in \mathcal{N}\}$ is the Lagrange basis of $\mathbb{V}$ [6].

We stress that $\mathcal{V}, \mathcal{N}, h, \Pi, I_{\mathsf{SZ}}, I_{\mathsf{L}}$, etc. do depend on the underlying triangulation $\mathcal{T}$, i. e., $\mathcal{V} = \mathcal{V}(\mathcal{T})$, $\mathcal{N} = \mathcal{N}(\mathcal{T})$, $h = h_{\mathcal{T}}$, $\Pi = \Pi_{\mathcal{T}}$, etc. We omit this dependence in the notation to increase readability.

2.2. **The proof of H^1-stability.** Assuming a suitable mesh-grading we prove H^1-stability of the L_2-orthogonal projection extending the technique presented in [4] to polynomial degrees $p > 1$.

Assumption 2.1. We assume that for $\mathcal{T}$ and $p \in \mathbb{N}$ there nodal values $\{h_z\}_{z\in\mathcal{V}}$ satisfying the following two conditions:

(1) There are constants $0 < c_0 \le C_0$ such that

$$c_0 h_T \le \min_{z\in\mathcal{V}(T)} h_z \quad \text{and} \quad \max_{z\in\mathcal{V}(T)} h_z \le C_0 h_T \qquad \forall\, T \in \mathcal{T}. \tag{2.2a}$$

i. e., $H \eqsim h_T$ on T.

(2) Let $H_+, H_- \in \mathbb{V}(\mathcal{T}, 1)$ be the piecewise linear functions with nodal values $H_+(z) = h_z$ and $H_-(z) = h_z^{-1}$, $z \in \mathcal{V}$. There is a constant $c_1 > 0$ such that

$$c_1\|V\|_T^2 \le \int_T I_{\mathsf{L}}(H_+V)\, I_{\mathsf{L}}(H_-V)\, dV \qquad \forall\, T \in \mathcal{T}. \tag{2.2b}$$

In the remaining of this section we suppose that Assumption 2.1 is valid. In Section 3 below we verify this assumption in 2d for a sequence of adaptively generated grids and moderate polynomial degree p.

Lemma 2.2. *For all $V \in \mathbb{V}$ we have*

$$\|h^{-1}V\|_\Omega \lesssim \|I_{\mathsf{L}}(H_-V)\|_\Omega,$$
$$\|hI_{\mathsf{L}}(H_-V)\|_\Omega \lesssim \|V\|_\Omega$$

and

$$c_1\|V\|_\Omega^2 \le \int_\Omega I_{\mathsf{L}}(H_+V)\, I_{\mathsf{L}}(H_-V)\, dV \qquad \forall\, T \in \mathcal{T}.$$

Proof. The third claim is a direct consequence of (2.2b) in combination with the additivity of $\|\cdot\|_\Omega^2$ and the integral.

To show the first and second claims, it obviously suffices to prove them for a single simplex T. We observe that $V \mapsto h_T\|I_{\mathsf{L}}(H_-V)\|_T$ defines a norm on $\mathbb{P}_p(T)$. Consequently, the equivalence of norms on $\mathbb{P}_p(T)$ in combination with (2.2a) and scaling arguments yields

$$\|V\|_T \lesssim h_T\|I_{\mathsf{L}}(H_-V)\|_T \le \|V\|_T \qquad \forall\, V \in \mathbb{P}_p(T).$$

This finishes the proof. □

We next derive a stability estimate for Π in a mesh-dependent norm.

Proposition 2.3 (Improved stability)**.** *We have*

$$\|h^{-1}\Pi u\|_\Omega \le \|h^{-1}u\|_\Omega \qquad \forall\, u \in L_2(\Omega).$$

Proof. For given $u \in H_D^1(\Omega)$ we define $V := I_{\mathsf{L}}(H_+^{-1}\Pi u) \in \mathbb{V}$. Comparing the nodal values reveals $I_{\mathsf{L}}(HV) = \Pi u$. Using Lemma 2.2 and the definition of Π we therefore deduce

$$\begin{aligned} c_1\|V\|_\Omega^2 &\le \int_\Omega I_{\mathsf{L}}(H_+V)I_{\mathsf{L}}(H_-V)\, dV = \int_\Omega \Pi u I_{\mathsf{L}}(H_-V)\, dV \\ &= \int_\Omega uI_{\mathsf{L}}(H_-V)\, dV \le \|h^{-1}u\|_\Omega\|hI_{\mathsf{L}}(H_-V)\|_\Omega \lesssim \|h^{-1}u\|_\Omega\|V\|_\Omega. \end{aligned}$$

where we have used Lemma 2.2 in the last step. Utilizing this lemma once more we finally arrive at

$$\|h^{-1}\Pi u\|_\Omega \lesssim \|I_{\mathsf{L}}(H_-\Pi u)\|_\Omega = \|V\|_\Omega \lesssim \|h^{-1}u\|_\Omega.$$ □

This brings us in position to prove the main result of this section.

Theorem 2.4 (H^1-stability)**.** *Suppose that $\mathcal{T}$ and p satisfy Assumption 2.1. Then the L_2-orthogonal projection $\Pi\colon H_D^1(\Omega) \to \mathbb{V}(\mathcal{T}, p)$ is H^1-stable and satisfies*

$$\|h^{-1}(\Pi u - u)\|_\Omega + \|\nabla\Pi u\|_\Omega \lesssim \|\nabla u\|_\Omega \qquad \forall\, u \in H_D^1(\Omega).$$

The hidden constant solely depends on c_0, C_0, c_1, p, and the shape regularity coefficient of $\mathcal{T}$.

Proof. Let $u \in H^1_D(\Omega)$ be given. We recall the inverse estimate $\|\nabla V\|_\Omega \lesssim \|h^{-1}V\|_\Omega$ for any $V \in \mathbb{V}$. Resorting to the Scott-Zhang interpolant we observe $\Pi I_{\mathsf{SZ}} u = I_{\mathsf{SZ}} u$ and set $e := I_{\mathsf{SZ}} u - u$. Proposition 2.3 then implies

$$\begin{aligned}\|\nabla(\Pi u - u)\|_\Omega &\le \|\nabla\Pi(u - I_{\mathsf{SZ}} u)\|_\Omega + \|\nabla(I_{\mathsf{SZ}} u - u)\|_\Omega \\ &\lesssim \|h^{-1}\Pi e\|_\Omega + \|\nabla e\|_\Omega \lesssim \|h^{-1} e\|_\Omega + \|\nabla e\|_\Omega \lesssim \|\nabla u\|_\Omega,\end{aligned}$$

where we have used (2.1) in the last step. This yields $\|\nabla \Pi u\|_\Omega \lesssim \|\nabla u\|_\Omega$. Finally, Proposition 2.3 and (2.1) conclude the proof by

$$\|h^{-1}(\Pi u - u)\|_\Omega \le \|h^{-1}\Pi e\|_\Omega + \|h^{-1} e\| \lesssim \|h^{-1} e\|_\Omega \lesssim \|\nabla u\|_\Omega. \qquad \square$$

3. H^1-Stability on Adaptively Generated Meshes

We next verify Assumption 2.1 for any refinement of some given conforming and exact triangulation $\mathcal{T}_0$ of a bounded polygon $\Omega \subset \mathbb{R}^2$ and moderate polynomial degree $p \in \mathbb{N}$. This in turn implies H^1-stability of the L_2-orthogonal projection Π by Theorem 2.4.

We suppose that $\mathcal{T}_0$ meshes the Dirichlet part $\partial_D\Omega$ of $\partial\Omega$ exactly. We denote by $\mathbb{T}$ the class of all conforming refinements of $\mathcal{T}_0$ generated either by the Newest Vertex Bisection (NVB), compare with [11], [2], [15] or [13] , the Red-Blue-Green Refinement (RBG), compare with [5], or the Red-Green Refinement (RG), compare with [8], of $\mathcal{T}_0$. In many cases the initial grid for NVB satisfies the following property.

Assumption 3.1 (Reflected Neighbours Condition for NVB)**.** Suppose $T, T' \in \mathcal{T}_0$ are direct neighbors with common edge $T \cap T' = E \in \mathcal{E}_0$. Then either E is the common refinement edge of both T and T', or E is neither the refinement edge of T nor of T'.

3.1. **Regularized mesh-size function.** For a given $\mathcal{T} \in \mathbb{T}$ we next introduce the regularized mesh-size function $H \in \mathbb{V}(\mathcal{T}, 1)$. We refer to [8, §3.1] for the detailed definition. The distance $\operatorname{dist}(z, z') \in \mathbb{N}_0$ of two vertices $z, z' \in \mathcal{V} = \mathcal{V}(\mathcal{T})$ is the minimal number of edges needed to connect z and z'. The distance of a vertex $z \in \mathcal{V}$ to an element $T \in \mathcal{T}$ is $\operatorname{dist}(z, T) := \min\{\operatorname{dist}(z, z') \mid z' \in \mathcal{V}(T)\}$. We let $\operatorname{gen}(T) \in \mathbb{N}_0$ be the generation of $T \in \mathcal{T}$ such that $h_T^2 = |T| \eqsim 2^{-\operatorname{gen}(T)}$. For a suitable penalty parameter $\mu \in \mathbb{N}$ we then define the nodal values of H by

$$H(z) = h_z := \min\{2^{(\mu \operatorname{dist}(z,T) - \operatorname{gen}(T))/2} \mid T \in \mathcal{T}\} \qquad \forall\, z \in \mathcal{V}. \tag{3.1}$$

We next resort to the following result from [8].

Lemma 3.2 (Mesh grading)**.** *For any adaptive grid $\mathcal{T} \in \mathbb{T}$ there exist $\mu \in \mathbb{N}$ such that the regularized mesh-size function defined by* (3.1) *satisfies for all $T \in \mathcal{T}$ the estimates*

$$\max_{z,z' \in \mathcal{V}(T)} \frac{h_z}{h_{z'}} \le \gamma = 2^{\frac{\mu}{2}}, \tag{3.2a}$$

$$c_0 h_T \le \min_{z \in \mathcal{V}(T)} h_z \quad \text{and} \quad \max_{z \in \mathcal{V}(T)} h_z \le C_0 h_T. \tag{3.2b}$$

The constants $0 < c_0 \le C_0$ solely depend on $\mathcal{T}_0$. In the NVB case with $\mathcal{T}_0$ satisfying Assumption 3.1 we have $\mu = 2$, which yields $\gamma = 2$. This is the best possible grading constant. In the NVB case with $\mathcal{T}_0$ violating Assumption 3.1 as well as in the RBG case we have $\mu = 3$, which results in $\gamma = 2^{3/2}$.In the RG case we have $\mu = 4$, which results in $\gamma = 4$. In any case we find $h_z^{-2} \in \mathbb{N}_0$ for all $z \in \mathcal{V}$.

Proof. Compare with [8, Theorem 3.1 and §5] for NVB and RG and [5, Proposition 4.1] and [8, Remark 5.5] for RBG. The final claim $h_z^{-2} \in \mathbb{N}_0$ is a direct consequence of the definition (3.1) of H. $\square$

3.2. **Verification of Assumption 2.1.** We next verify Assumption 2.1 for the regularized mesh-size function H introduced above and moderate polynomial degree p. The estimate (3.2b) of Lemma 3.2 is (2.2a), which is the first part.

We observe that (2.2b) is invariant under any affine transformation. Thus we can fix any triangle T and it suffices to show for this selected triangle the estimate

$$c_1 \|V\|_T^2 \leq \int_T I_{\mathsf{L}}(H_+ V)\, I_{\mathsf{L}}(H_- V)\, dV \qquad \forall V \in \mathbb{P}_p, \tag{3.3}$$

where $I_{\mathsf{L}} \colon C^0(T) \to \mathbb{P}_p$ is the Lagrange interpolant on T. Moreover, since $h_z^{-2} \in \mathbb{N}_0$ we can rescale H in (3.3) such that we can assume

$$\max\{h_z \mid h_z \in T\} = 1. \tag{3.4}$$

We next convert (3.3) into a generalized eigenvalue problem such that the minimal eigenvalue constant $\lambda_{\min}$ is the optimal constant c_1 in (3.3). Let $\{\Phi_1, \ldots \Phi_N\}$ be the Lagrange-Basis of $\mathbb{P}_p$ attached to the Lagrange grid $\{a_1, \ldots, a_N\}$ on T. Let $\boldsymbol{M}$ be the mass matrix

$$\boldsymbol{M}_{nm} = \int_T \Phi_n \Phi_m \, dV, \qquad n, m = 1, \ldots N,$$

and let $\boldsymbol{A}^H$ be the disturbed mass matrix

$$\boldsymbol{A}^H_{nm} = \frac{1}{2} \int_T I_{\mathsf{L}}(H_+ \Phi_n) I_{\mathsf{L}}(H_- \Phi_m) + I_{\mathsf{L}}(H_+ \Phi_m) I_{\mathsf{L}}(H_- \Phi_n)\, dV, \qquad n, m = 1, \ldots N.$$

The property $\Phi_n(a_m) = \delta_{nm}$ in combination with the definition of I_{L} then readily yields

$$\boldsymbol{A}^H_{nm} = \frac{1}{2}\Big(H_+(a_n) H_-(a_m) + H_+(a_m) H_-(a_n)\Big) \boldsymbol{M}_{nm}, \qquad n, m = 1, \ldots N.$$

Lemma 3.3 (Eigenvalue problem)**.** *If the minimal eigenvalue $\lambda_{\min}$ of the generalized eigenvalue problem*

$$\boldsymbol{A}^H \boldsymbol{x} = \lambda \boldsymbol{M} \boldsymbol{x} \tag{3.5}$$

is positive, then (2.2b) *is valid with constant $c_1 = \lambda_{\min} > 0$.*

Proof. For $V \in \mathbb{P}_p$ set $\boldsymbol{v} := [v_n]^\top_{n=1,\ldots,N} = [V(a_n)]^\top_{n=1,\ldots,N}$. From $V = \sum_{n=1}^N v_n \Phi_n$ we deduce

$$\int_T I_{\mathsf{L}}(H_+ V)\, I_{\mathsf{L}}(H_- V)\, dV = \boldsymbol{v}^\top \boldsymbol{A}^H \boldsymbol{v} \geq \lambda_{\min} \boldsymbol{v}^\top \boldsymbol{M} \boldsymbol{v} = \lambda_{\min} \|V\|_T^2. \qquad \square$$

$\mu = 1$

m_0	m_1	m_2
0	0	1
0	1	1

$\mu = 2$

m_0	m_1	m_2
0	0	2
0	1	2
0	2	2

$\mu = 3$

m_0	m_1	m_2
0	0	3
0	1	3
0	2	3
0	3	3

$\mu = 4$

m_0	m_1	m_2
0	0	4
0	1	4
0	2	4
0	3	4
0	4	4

TABLE 1. $h_z \in \{2^{-m/2} \mid 0 \leq m \leq \mu\}$ for different values of μ.

$\mu = 1$	
p	$\lambda_{\min}$
1	0.9171366643508360
2	0.9516692464539273
3	0.9641852703796995
4	0.9682402611804647

$\mu = 2$	
p	$\lambda_{\min}$
1	0.6584936490538895
2	0.8008134821910052
3	0.8523960262167701
4	0.8691079422969096

$\mu = 3$	
p	$\lambda_{\min}$
1	0.1926922946340332
2	0.5291308340629023
3	0.6510699580030909
4	0.6905762763645189

$\mu = 4$	
p	$\lambda_{\min}$
1	−0.5367785792574971
2	0.1036606698595243
3	0.3357821179754649
4	0.4109857403360982

TABLE 2. Minimal eigenvalues $\lambda_{\min}$ of (3.5) for different penalty parameters $\mu = 1, \dots, 4$.

The grading condition (3.2a), the property $h_z^{-2} \in \mathbb{N}_0$, in combination with the normalization (3.4) implies that H can only attain a finite number of discrete values at the vertices of T, namely

$$\{h_z^2 \mid z \in \mathcal{V}(T)\} \subset \{2^{-m} \mid m = 0, \dots, \mu\}.$$

This means, for given polynomial degree p we only have to study the eigenvalue problem (3.5) for a small number of possible affine function H on T. The possible nodal values of H are reported in Table 1 for penalty parameters $\mu = 1, \dots, 4$. For $p = 1, \dots, 4$ we have determined numerically these minimal eigenvalues. They are reported in Table 2. We see $\lambda_{\min} > 0$ for $p = 1, \dots, 4$ if $\mu \le 3$. For $\mu = 4$ we still have $\lambda_{\min} > 0$ for $p = 2, 3, 4$ but not for $p = 1$. We can then present the main result in the general case.

Theorem 3.4 (H^1-stability)**.** *Let $\mathcal{T}_0$ be any initial, conforming triangulation of Ω that meshes $\partial_D\Omega$ exactly. Let $\mathbb{T}$ be the class of all conforming refinements of $\mathcal{T}_0$ generated by either NVB or RBG, then the L_2-orthogonal projection is H^1-stable for all $\mathcal{T} \in \mathbb{T}$ and polynomial degrees $p \le 4$. If instead $\mathbb{T}$ is the class of all conforming refinements of $\mathcal{T}_0$ generated by RG then the L_2-orthogonal projection is H^1-stable for all $\mathcal{T} \in \mathbb{T}$ and polynomial degrees $p = 2, 3, 4$.*

We see then that in the case of NVB and RBG the missing case $p = 2$ has been resolved. Moreover, for RG the stability for $p = 2, 3, 4$ is a completely new result.

REFERENCES

[1] BANK, R. E., AND YSERENTANT, H. On the H^1-stability of the L_2-projection onto finite element spaces. *Numerische Mathematik 126*, 2 (2014), 361–381.

[2] BÄNSCH, E. Local mesh refinement in 2 and 3 dimensions. *IMPACT Comput. Sci. Engrg. 3* (1991), 181–191.

[3] BRAMBLE, J. H., PASCIAK, J. E., AND STEINBACH, O. On the stability of the L_2 projection in $H^1(\Omega)$. *Math. Comput. 71*, 237 (2002), 147–156.

[4] CARSTENSEN, C. Merging the Bramble-Pasciak-Steinbach and the Crouzeix-Thomée criterion for H^1-stability of the L_2-projection onto finite element spaces. *Math. Comp. 71*, 237 (2002), 157–163.

[5] CARSTENSEN, C. An adaptive mesh-refining algorithm allowing for an H^1 stable L_2 projection onto Courant finite element spaces. *Constr. Approx. 20*, 4 (2004), 549–564.

[6] CIARLET, P. G. *The finite element method for elliptic problems*, vol. 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[7] Crouzeix, M., and Thomée, V. The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. *Math. Comp. 48*, 178 (1987), 521–532.

[8] F. D. Gaspoz, C.-J. Heine, and K. G. Siebert, Optimal grading of the newest vertex bisection and H^1-stability of the L_2-projection. *IMA J. Numer. Anal. 36* (2016), 1217–1241.

[9] Karkulik, M., Pavlicek, D., and Praetorius, D. On 2D newest vertex bisection: optimality of mesh-closure and H^1-stability of L_2-projection. *Constr. Approx. 38*, 2 (2013), 213–234.

[10] Maubach, J. M. Local bisection refinement for n-simplicial grids generated by reflection. *SIAM J. Sci. Comput. 16*, 1 (1995), 210–227.

[11] Mitchell, W. F. *Unified Multilevel Adaptive Finite Element Methods for Elliptic Problems.* PhD thesis, Department of Computer Science, University of Illinois, Urbana, 1988.

[12] Scott, L. R., and Zhang, S. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp. 54*, 190 (1990), 483–493.

[13] Stevenson, R. The completion of locally refined simplicial partitions created by bisection. *Math. Comp. 77*, 261 (2008), 227–241.

[14] Tantardini, F., and Veeser, A. The L_2-projection and quasi-optimality of Galerkin methods for parabolic equations. *SIAM J. Numer. Anal. 54*, 1 (2016), 317–340.

[15] Traxler, C. T. An algorithm for adaptive mesh refinement in n dimensions. *Computing 59* (1997), 115–137.

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